

**NASA
Reference
Publication
1222**

April 1990

Foundations of Measurement and Instrumentation

Isidore Warshawsky

(NASA-RP-1222) FOUNDATIONS OF MEASUREMENT
AND INSTRUMENTATION (NASA) 232 p. CSCL 147

NO-11551

MI/35 Unclass
077343

NASA

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NASA

National Aeronautics and
Space Administration
Office of Management
Scientific and Technical
Information Division

Preface

This text is intended to provide the user of instrumentation with an understanding of the factors that influence instrument selection, instrument application, the interpretation of results, and the subsequent planning of future operations. In selecting the tools and techniques of measurement, consideration must be given to factors like their reliability, accuracy, speed, cost, and role in contributing to the ultimate goal of the measurement operation. Because the relative importance of these factors varies with each application, the balance among them requires a human judgment. A mathematical formula may be used to describe this judgment, but only after the judgment has been made.

The text also seeks to provide that intuitive understanding of the capabilities and limitations of measurement methods that will permit avoidance of an excessive precision and detail beyond that needed by a practical engineering problem. Such matching of effort to need not only saves time and labor, but also attests to the soundness and maturity of the judgment that has been exercised.

The chain of variables, components, or steps that comprise a measurement operation can proceed from the physical variable to be measured, to a sensor and its installation, to transmission of the signal from the sensor to a signal transducer, conditioner, or modifier, to an indicator or recorder, to data analysis or manipulation, and to the interpretation and understanding of the data. This text, being directed to the instrument user rather than to the instrument designer, treats, in categorical terms, only those items in the chain that principally control the accuracy, reliability, and utility of the information acquired; these items are the first two or three items of the chain, and the very last item. The quality, sophistication, and pace of development of modern instruments for data acquisition and manipulation are such that the intermediate items of the chain rarely limit the ability to acquire and to understand the information that is desired about the physical variable being studied.

This material is the substance of a set of thirty 2-hour lectures presented to members of the senior engineering staff of what is currently known as the NASA Lewis Research Center. The subjects selected for treatment in this text are those which have been found to be most significant in the practical application of instruments and of measurement techniques by this staff. The more basic principles are presented in the main text of each chapter; additional details appear as appended notes following each chapter. Illustrative examples are chosen, as far as possible, to represent practical applications, sometimes in several engineering disciplines. Many of these examples are in a form that makes them suitable for use as problems in a formal course of instruction.

An understanding of the language of the differential calculus is necessary; but facility in its operations usually is not. To facilitate cross reference, and the text's use as a handbook, figures and specialized tables bear the same number as the section in which they appear.

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CHAPTER 1. DIMENSIONS

1.0. Physical quantities. Symbols. The development of physics has included the identification of measurable physical quantities representing properties or characteristics of matter, and the establishment of relationships among these quantities. Each newly identified quantity has been given a name that is usually widely accepted. [Note N1.0] For convenience, each quantity is also assigned a symbol, although this assignment is arbitrary and may vary with the discipline in which the quantity is used, may vary from one text to another, and may even vary among different sections of the same text.

The subject of dimensions deals with the names of these quantities. Each physical quantity, property, or attribute may be assigned a *dimension* characterized by its name, which is descriptive of its nature or behavior. For example, mass, energy, and temperature may be said to have the dimensions of mass, energy, and temperature, respectively. As a separate matter, the term “unit” characterizes the quantitative magnitude of some physical quantity, property, or attribute. For example, the gram, slug, and pound are all units of mass; the joule, erg, and British thermal unit are all units of energy; the kelvin, degree Celsius, and degree Fahrenheit are all units of temperature.

The ratio between two quantities having the same dimensions is said to have the dimension of unity. The ratio itself is said to be *dimensionless* or *nondimensional*. All numerics (pure numbers like $\sqrt{2}$ or π) and transcendental functions (like the logarithm or the sine) are dimensionless.

The concept of dimensions was introduced by Fourier in 1822. (Ref. 1-1) It facilitates the qualitative development and understanding of the laws of physics even before they are converted to quantitative relationships through choice of the units of measurement.

If each physical quantity is considered to have a *dimension*, it is found that the dimensions of any physical quantity can be described as a combination of some or all members of a group of five independent dimensions. The selection of which five quantities are to be used in the group is arbitrary, subject only to the condition that each be independent of the others (i.e., that none shall be merely a power or product of the others) and is generally made on the basis of convenience. The term “independent dimension” is analogous to the term “independent variable” in mathematics.

In this chapter, square brackets around the symbol for a quantity are used to signify the dimensions of that quantity. Thus, [A] is read “the dimensions of A.” The dimension itself is represented by an unbracketed symbol for that quantity; in this text, that symbol will always be capitalized. The symbol for the quantity itself may or may not be capitalized. Thus, if area a and length l are two physical quantities, and A and L represent the dimensions *area* and *length*, respectively,

$$[a] = A = L^2; [l] = A^{1/2} = L; [a/l] = L$$

1.1 Relationships among physical quantities. In most cases, the basic laws of physics are statements of proportionalities between physical quantities or powers of those quantities. [Note N1.1] The conversion of proportionalities into equalities requires the assignment of a factor of proportionality. This factor may itself have dimensions or, if it is dimensionless, may be a pure number (a numeric) or the ratio of two physical quantities that have the same dimensions; the factor also depends on the units in which the physical quantities are measured. The subject of units will be treated in the next chapter.

Tables 1-1 to 1-5¹ list some quantities of physics and their dimensional representations in a useful grouping of independent dimensions. Some tables are arranged to show the parallelism between analogous quantities. More complete tables are available in handbooks of physics and engineering. In the column which lists definitions of symbols by a formula, the required proportionality sign has been replaced by a proportionality factor (often unity) in a manner that provides a fully consistent set of equations for those systems of units that have had wide acceptance since 1890. The selection of the value of the proportionality factor will be treated in the next chapter, but it is important to note that the dimensions of a physical quantity do not depend on its magnitude and are independent of the units in which it is measured.

Example 1.1. Names, symbols, and proportionalities are illustrated by the following relationships among physical quantities:

- | | | |
|----------|--|----------------|
| (1.1-1a) | (force F) \propto (mass M) \times (acceleration a) | Newton's law |
| (1.1-1b) | (electrostatic flux Ψ) \propto (electric charge Q) | Gauss's law |
| (1.1-1c) | (magnetic flux Φ) \propto (magnetic pole strength m^*) | Gauss's law |
| (1.1-1d) | (magnetomotive force \mathcal{F}) \propto (current I) \times (turns n) | solenoidal law |
| (1.1-1e) | (force F) \propto (current I) \times (induction B) \times (length L) | |
| (1.1-1f) | (energy change ΔW) \propto (heat capacity C) \times (temperature change $\Delta\theta$) | |

All of the above relationships happen to require a dimensionless factor of proportionality. When such a relationship among physical quantities is converted into the corresponding relationship among their dimensions, the proportionality sign becomes an equality sign. Thus,

- | | |
|----------|---------------------------------|
| (1.1-2a) | $[F] = [M] [a]$ |
| (1.1-2b) | $[\Psi] = [Q]$ |
| (1.1-2c) | $[\Phi] = [m^*]$ |
| (1.1-2d) | $[\mathcal{F}] = [I] [n] = [I]$ |
| (1.1-2e) | $[F] = [I] [B] [L]$ |
| (1.1-2f) | $[W] = [C] [\theta]$ |

The following relationships require a dimensional factor of proportionality. The corresponding relationship among dimensions may not be written as an equality; it serves merely to define the dimensions of the factor of proportionality.

- | | | |
|----------|--|--------------|
| (1.1-3g) | (radiant flux Φ_r) \propto (temperature θ) ⁴ \times (area A) | Stefan's law |
| (1.1-3h) | (energy change ΔW) \propto (frequency change $\Delta\nu$) | Planck's law |

¹For ease of later reference, Tables 1-1 through 1-5 appear at the end of this text on pages 208 to 213.

$$(1.1-3i) \quad (\text{pressure } p) \propto (\text{density } \rho) \times (\text{temperature } \theta) \quad \text{Gas law}$$

$$(1.1-3j) \quad (\text{mass electrodeposited } m) \propto (\text{total charge } Q) \quad \text{Faraday's law}$$

Thus, the respective factors of proportionality have the following dimensions:

$$(1.1-4g) \quad [\Phi_e]/[\theta^4 A] = W/(L^2 T \theta^4) \quad (\text{Stefan-Boltzmann constant, } \sigma)$$

$$(1.1-4h) \quad [\Delta W]/[\Delta \nu] = WT \quad (\text{Planck's constant, } h)$$

$$(1.1-4i) \quad [p]/[\rho \theta] = W/(M \theta) \quad (\text{specific gas constant, } R)$$

$$(1.1-4j) \quad [m]/[Q] = IRT^2/L^2 \quad (\text{Faraday constant, } F)$$

1.2. Psychophysical quantities. These are quantities that represent the perception of physical quantities by a human observer, often called the *standard observer*.

In illumination engineering, psychophysical quantities are based on the spectral sensitivity of the human eye. The word “luminous” is prefixed to the the name of the quantity to denote the psychophysical quantity, in contrast to the prefix “radiant” for the purely physical quantity. (*Example: luminous flux versus radiant flux.* A source emitting considerable flux in the deep ultraviolet emits no luminous flux because the radiation is invisible.)

In audiometry, psychophysical quantities are based on the auditory sensitivity of the human ear or on human psychological characteristics. (*Examples: loudness, noisiness.*) The commonly used psychophysical quantities happen to be dimensionless because they are functions of the ratios of two similar physical quantities, like pressure or power.

1.3. The Pi theorem. Dimensionless numbers. If n independent physical quantities enter into some physical relation, and the dimensional expressions for these quantities involve a total of k independent fundamental dimensions (note that $k \leq 5$), the relation among the physical quantities can be expressed as a function of $n-k$ independent dimensionless products of these n quantities in the form

$$(1.3-1) \quad f(\Pi_1, \Pi_2, \Pi_3, \dots, \Pi_{n-k}) = 0$$

where Π_i is a dimensionless product of a number of the n physical quantities. The word “product” includes the operation of raising any one quantity to some power. This is currently called the Pi theorem.²

The scaling of models involves changing the magnitudes of the quantities that enter into any Π_i without changing the magnitude of Π_i itself. Similarly, calibration of an instrument needs to be performed only over a range of values of each Π_i , rather than over a range of values of each quantity that enters into any Π_i ; calibration effort is, thereby, reduced.

In any one discipline, certain dimensionless products recur frequently. They are then termed *dimensionless numbers* and are often named after a scientist prominent in that discipline. The General Conference on Weights and Measures (CGPM) recommends that they be written as N_{Xr} , where the subscript identifies the scientist. Where the meaning is unambiguous, they have also been written as Xr rather than as subscripted numbers. (*Examples: Reynolds number, N_{Re} or Re ; Prandtl number, N_{Pr} or Pr).*

Table 1-6³ contains a listing of a few dimensionless numbers. More extensive compilations have been published (Refs. 1-4 and 1-5).

²The theorem was stated by Riabouchinsky in 1911 (Ref. 1-2) for the case $k = 3$, in application to problems in mechanics. It was recognized independently by Buckingham and stated for the more general case $k \geq 3$, in 1914 (Ref. 1-3).

³For ease of later reference, Table 1-6 appears at the end of this text on page 214.

1.4. The Pi-zero theorem. The Pi theorem leads to the corollary that a physical quantity y can be expressed as the product π_0 of other physical quantities (the dimensions of π_0 being the same as the dimensions of y) multiplied by a function of dimensionless products of the quantities entering into the physical relationship. For example, suppose that y occurs to the m^{th} power in the product Π_1 of Eq. (1.3-1). Then we take $\pi_0 = y \Pi_1^{-1/m}$ and rewrite Eq. (1.3-1) in the form

$$(1.4-1) \quad y = \pi_0 F(\Pi_2, \Pi_3, \dots, \Pi_{n-k})$$

This relation leads to a fundamental law of measurement: The percentage accuracy of measuring y can be no better than the percentage accuracy of measuring π_0 . (See Sec. 3.28.)

A restatement of this law is that to obtain a given accuracy in the measurement of a quantity y , it is necessary (but not always sufficient) to compare it with another quantity π_0 that is known to at least the same percentage accuracy, π_0 being a product of variables such that the dimensions of π_0 are the same as the dimensions of y .

A particularly simple case, very common in instrument calibration, is $n-k = 1$, so that there is only one product Π_1 . Then the unknown quantity y can be expressed as a simple product of powers of the other variables, and the accuracy of knowledge of the unknown is equal to the accuracy of the product, if the calibration or comparison procedure itself introduces negligible error.

In the simplest case, π_0 may be a single physical quantity having the same dimensions as y .

Example 1.4a. "Potentiometer of the first kind." (Fig. 1.4(a)) To measure an unknown electromotive force (emf) e_x , the current i through a resistor r_s is first adjusted until the voltage drop across r_s is equal to a known standard emf e_s , derived from a *standard cell* or its equivalent (e.g., a Zener-diode-based power supply), as indicated by a null detector. Then e_s is replaced by e_x and r_x is adjusted until the voltage drop across r_x is equal to e_x . Then $e_x = e_s(r_x/r_s)$. The value of i need not be known but must have remained constant during the entire operation. Here $\pi_0 = e_s$. The percentage accuracy of knowledge of e_x is no better than the percentage accuracy of knowledge of e_s . The ratio r_x/r_s must be known with negligible error, but the absolute value of the resistances need not be known.

Example 1.4b. "Potentiometer of the second kind." (Fig. 1.4(b)) The current i through a fixed resistor r_s is adjusted until the voltage drop across r_s is equal to an unknown emf e_x , as indicated by a null detector. The current i is measured. Here $\pi_0 = ir_s$. The percentage accuracy of knowledge of e_x is no better than the percentage accuracy of knowledge of the product ir_s .

This arrangement lends itself to the use of a feedback amplifier to replace a manual operator and to the use of an oscillograph to replace the ammeter, in order to achieve rapid emf measurement while almost maintaining the null potentiometer feature.

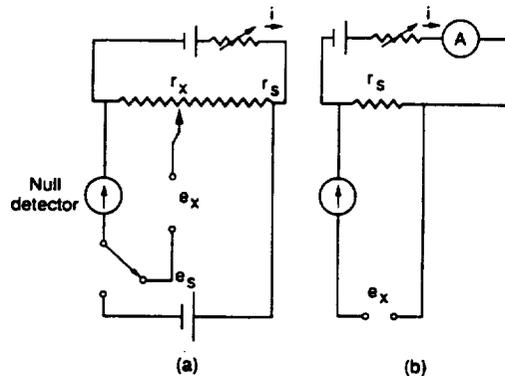


Figure 1.4 - Null-potentiometer circuits.

Example 1.4c. The value r_x of a resistor is measured, using a Wheatstone bridge with ratio arms r_A and r_B , by comparing it with a standard resistor r_s . Here $\pi_0 = r_s$. The percentage accuracy of knowledge of r_x can be no better than the percentage accuracy of knowledge of r_s and of the ratio r_A/r_B .

Example 1.4d. The value r of a resistor is measured by determining the voltage drop e across the resistor when a measured current i passes through it. Here, $\pi_0 = e/i$. The percentage accuracy of knowledge of r can be no better than the percentage accuracy of knowledge of the ratio of voltage drop to current.

Example 1.4e. The period T of oscillation of a pendulum of length L swinging in a plane with moderate amplitude to a maximum angle α under a gravitational acceleration g is such that

$$(1.4-2) \quad T \propto (L/g)^{1/2} \left\{ 1 + (1/4)\sin^2(\alpha/2) \right\}$$

Here, $\pi_0 = (L/g)^{1/2}$. The remainder of the expression is a small correction factor whose application is usually warranted, although the uncertainty in this factor usually contributes negligibly to the inaccuracy of measuring T .

Example 1.4f. The power loss P from a section Δx of a substantially infinite wire of diameter d when it is transverse to a gas stream of velocity v , density ρ , thermal capacity c , and thermal conductivity k is given by

$$(1.4-3) \quad P \propto k \cdot \Delta x \cdot \Delta \theta \left\{ 1 + (2\pi v \rho c d / k)^{1/2} \right\}$$

when it is heated so that a temperature difference $\Delta \theta$ exists between wire and gas. Here, $\pi_0 = k \cdot \Delta x \cdot \Delta \theta$ and $\Pi_2 = v \rho c d / k$. (Π_2 is the Peclet number.) At low velocities, Π_2 contributes moderately to the ratio $P / (k \cdot \Delta x \cdot \Delta \theta)$, and the uncertainty in knowledge of Π_2 contributes negligibly to the ratio.⁴

Example 1.4g. The mass flow rate \dot{m} of the gas through a restriction of area A is determined by measuring total temperature T_t , static pressure p_s , and the difference Δp between total pressure p_t and static pressure. The applicable relationship is $\dot{m} \propto C A b \{ 2p_s \cdot \Delta p / (RT_t) \}^{1/2}$ where C , the dimensionless discharge coefficient, is principally a function of restriction shape (and secondarily of the Reynolds number), R is the specific gas constant, and b is a dimensionless function (whose magnitude differs less than 10 percent from unity) of the specific heat ratio of the gas and of p_s/p_t . Here

$$(1.4-4) \quad \pi_0 = A \left\{ 2p_s \cdot \Delta p / (RT_t) \right\}^{1/2}$$

and $\Pi_2 = C$, $\Pi_3 = b$. The inaccuracy of measuring \dot{m} is determined principally by the inaccuracy of measuring π_0 . A slight contribution to inaccuracy originates from the usual uncertainty in Π_2 , and a negligible contribution originates from the usual uncertainty in Π_3 .

An Operational Note. The Pi-zero theorem may impose severe limitations on the ability to measure the absolute value of a physical quantity. However, it does not inhibit the ability to measure the difference or the ratio between the unknown quantity y and a reference quantity π_0 which, though not adequately known in absolute magnitude, is known to be very stable. Thus, the null detectors used in Examples 1.4a to 1.4c may be able to detect emf or resistance differences smaller than the uncertainty in knowledge of the absolute value of π_0 .

⁴In the thermal-conductivity-type gas analyzer, the conductivity k is deduced from accurate measurements of P and $\Delta \theta$. The velocity v must be nonzero in order that changes in gas content may be followed.

Resistance-strain-gage bridges, as used for strain, load, or pressure measurement, routinely measure fractional resistance changes $\Delta R/R$ so small that the uncertainty δR in knowledge of the absolute value of R may be 1000 times ΔR .

1.5. Dimensional analysis. If a physical situation is represented by an equation, and if each quantity in the equation is replaced by its dimensional equivalent, the resulting dimensional equation must be satisfied. (In the process of substituting dimensions, a pure numeric is replaced by unity.) In a strict mathematical sense, this does not imply that each term of a sum must have the same dimensions. (Example: if $F = ma$ and $A = \pi r^2$, then $F + A = ma + \pi r^2$ and $[F] + [A] = [ma] + [\pi r^2]$.) However, in all practical situations, one does seek to sum forces, powers, energies, areas, voltages, etc. In such cases, a check on the dimensions of each term may be the first step in confirming the correctness of the summation.

If the physical relation is not known, but the identities of the physical quantities involved are believed to be known, it is sometimes possible to determine how the physical quantities should be multiplied together in order to describe the physical situation. Suppose that there are n physical quantities y_1, y_2, \dots, y_n that are known to affect the phenomenon being studied, and that all n quantities can be represented by a total of k dimensions ($k \leq 5$). To find one of the $n-k$ products Π_i that enter into Eq. (1.3-1), assign an unknown exponent a, b, \dots to each of the respective y_i and seek a product $y_1^a y_2^b \dots$ such that, when each y_i is replaced by its dimensional equivalent, the exponents of each dimension in the product will vanish, since Π_i is dimensionless.

Similarly, if a quantity y_1 is believed to depend on other quantities y_2, y_3, \dots one may be led to an appropriate π_0 by replacing each y_i with its dimensional equivalent $[y_i]$ and finding the exponents a, b, c, \dots that will satisfy the relation

$$(1.5-1) \quad [y_1] = [y_2]^a [y_3]^b \dots$$

Then

$$(1.5-2) \quad \pi_0 = y_2^a y_3^b \dots$$

Operations of the type described in the two preceding paragraphs are termed *dimensional analysis*. Dimensional analysis

- (a) will not determine the functional relationship between the $n-k$ products of Eqs. (1.3-1) or (1.4-1),
- (b) will not establish the value of any numerics that should multiply the respective products or be added to them, and
- (c) will not assure that the products so determined have any real significance in the physical situation being studied.

On the other hand, the results obtained by dimensional analysis may provide a guide to the conduct of experiments to verify that the products do have physical significance, thereby to establish numerical constants, and to establish the functional relationships among the various products. (If no product can be formed, the indication is that some physical variable has been overlooked.) Consequently, dimensional analysis is a tool whose effectiveness in solving any problem is determined *a priori* by the quality of physical intuition available and *a posteriori* by the amount of experimental evidence that is available.

Example 1.5a. The impact force F on a plate normal to an airstream is believed to be dependent on the plate area A , the air density ρ , and the air velocity v . Using the M,L,T system of dimensions, we have $n = 4$ and $k = 3$. The product $F^a A^b \rho^c v^d$ is converted into the dimensional form $(MLT^{-2})^a (L^2)^b (ML^{-3})^c (LT^{-1})^d$. Setting the exponents of M, L, and T equal to zero leads to the three simultaneous equations $a + c = 0$, $a + 2b - 3c + d = 0$, $-2a - d = 0$. Arbitrarily setting $a = 1$ yields $b = -1$, $c = -1$, $d = -2$. Hence, $f(\rho A v^2 / F) = 0$, or $F = \rho A v^2 \times$ (a numeric).

Example 1.5b. The period T of a pendulum is believed to be dependent on its length l , the mass m at its free end, and the acceleration of gravity g . Using the M,L,T system of dimensions, we have $n = 4$, $k = 3$. The product $T^a l^b m^c g^d$ is converted into the dimensional form

$(T)^a(L)^b(M)^c(LT^{-2})^d$. Setting the exponents of M, L, and T equal to zero leads to the three simultaneous equations $a - 2d = 0$, $b + d = 0$, $c = 0$. Arbitrarily setting $a = 2$ yields $b = -1$ and $d = 1$. Hence, $f(T^2g/L) = 0$ or $T^2 = (l/g) \times (\text{a numeric})$.

Example 1.5c. The resonant frequency f of an electrical circuit is believed to be dependent on the lumped values of inductance L , capacitance C , and resistance R of the circuit. All four of these quantities may be represented in terms of two dimensions: R and T. Thus, $n = 4$ and $k = 2$, so that two dimensionless products are to be sought. Converting $f^aL^bC^cR^d$ into its equivalent dimensional form yields fewer equations than unknowns. However, assuming the expression $f^aL^bC^c$ readily yields the dimensionless quantity f^2LC , and assuming the expression $L^aC^bR^c$ readily yields the dimensionless quantity R^2C/L . Thus,

$$F(f^2LC, R^2C/L) = 0$$

These two groupings offer several advantages: First, since f appears in only one of them, an explicit expression for f will be possible; second, since R appears in only one of them, the expression containing R will exclusively represent the effect of energy dissipation. Thus, one may write

$$f = (LC)^{-1/2} \phi(R^2C/L)$$

where the function ϕ is a numerical multiplier of $(LC)^{-1/2}$, which is also the π_0 of Sec. 1.4.

Example 1.5d. The power loss P from an electrically heated length Δx of a very thin, very long wire of diameter d , when it is transverse to a gas stream of velocity v , density ρ , specific heat c , thermal conductivity k , and viscosity η , and where a temperature difference $\Delta\theta$ exists between wire and gas, is to be examined for a likely functional relationship. In the dimensional system H, L, T, Θ , one finds $n = 9$ and $k = 4$, so that up to five dimensionless products are possible. It is reasonable to expect P to be proportional to Δx and to $\Delta\theta$. Since $[P/(\Delta x \cdot \Delta\theta)] = HL^{-1}T^{-1}\Theta^{-1}$, which is also $[k]$, we choose $P/(k \cdot \Delta x \cdot \Delta\theta)$ as Π_1 . (It is termed the Nusselt number, N_{Nu} .) If P , Δx , and $\Delta\theta$ are eliminated as being unlikely to appear elsewhere, n becomes 6, so that two more dimensionless products are to be sought. We choose the Reynolds number $N_{Re} = dv\rho/\eta$ as one of these because it is known to enter commonly into heat transfer phenomena in a flowing gas. For the other product, we solve

$$[\rho^a c^b k^c \eta^d] = (ML^{-3})^a (HM^{-1}\Theta^{-1})^b (HL^{-1}T^{-1}\Theta^{-1})^c (ML^{-1}T^{-1})^d = 1$$

which results in $a = 0$, $b = -1$, $c = 1$, and $d = -1$, so that $c\eta/k$ is Π_3 . (It is termed the Prandtl number, N_{Pr} .) Hence,

$$(1.5-3) \quad f\left(\frac{P}{k \cdot \Delta x \cdot \Delta\theta}, \frac{dv\rho}{\eta}, \frac{c\eta}{k}\right) = 0.$$

The first product represents the effect of the input electric power, the second represents the effect of gas stream motion, and the third represents the effect of intrinsic gas properties. Equation (1.5-3), when rewritten in the form

$$(1.5-4) \quad P = k \cdot \Delta x \cdot \Delta\theta F(N_{Re}, N_{Pr})$$

may be compared with Eq. (1.4-3) by noting that the Peclet number is $N_{Pe} = N_{Re}N_{Pr}$ (Footnote 5).

⁵This example is the basis of hot-wire anemometry. In its usual applications, the principal instrumental measurement is that of the ratio $P/(\Delta x \cdot \Delta\theta)$; any deviation of k from its nominal value is treated as a systematic correction. The principal independent variable is the Reynolds number; the Prandtl number may usually be assumed to remain constant.

1.6. Empirical coefficients. The unknown numerics that appear in the equations derived by dimensional analysis can sometimes be determined by further theoretical analysis; at other times, recourse must be had to purely empirical determinations; and at still other times, the value predicted by theory must be multiplied by an empirically determined coefficient as the result of experiment. Much of the work in the “calibration” of instruments consists of the determination of such empirical coefficients, often termed “calibration factors”. A complete calibration includes the determination of the value of the calibration factor over the full range of Π_i values that will be encountered during the use of the instrument.

Example 1.6a. In Example 1.5a, Bernoulli’s principle will indicate that the numeric should be $1/2$, but experiment will show that this numeric must be multiplied by an empirical coefficient C_D (the *drag coefficient*) that is principally a function of the Reynolds number N_{Re} and of the shape of the frontal surface of the plate.

Example 1.6b. In Example 1.5b, theoretical analysis can usually provide the numerics that will correct for the fact that any real pendulum is a compound one rather than a simple one. However, empirical determination is required of the correction for the ambient atmosphere, which contributes entrained mass and viscous damping. The Stokes number N_{Sk} may be the appropriate independent variable. A similar empirical correction also applies to tuning forks and to piezoelectric crystals vibrating in the air.

Example 1.6c. In Example 1.5d, experience shows that the form of $F(N_{Re}, N_{Pr})$ is $F = a + bN_{Re}^c N_{Pr}^d$ and that only empirical determination of the four constants a, b, c, d in the actual operating range of N_{Re} and N_{Pr} can yield the value of F to an acceptable degree of accuracy.

NOTES FOR CHAPTER 1

N1.0 Occasionally, the widely accepted name may not be semantically correct. For example, *molecular weight* means *relative molar mass* or “multiple of unit atomic mass”. (Since 1960, by international agreement among chemists and physicists, unit atomic mass has been $1/12$ of the mass of one atom of carbon 12.)

N1.1 For example,

- (a) the acceleration of a given mass is proportional to the force acting upon it (Newton’s law);
- (b) the flux emanating from a point source (of electric charge, magnetic charge, or radiation) is proportional to the strength of the source (Gauss’s law);
- (c) the magnetomotive force generated by a current passing through a coil is proportional to the magnitude of the current and to the number of turns in the coil (solenoidal law);
- (d) the force on a current-carrying conductor in a magnetic field is proportional to the magnitude of the current and to the density of the magnetic field (a consequence of Ampere’s law);
- (e) the energy required to raise the temperature of a given mass of a passive solid is proportional to the specific heat capacity of the material and to the magnitude of the temperature change.
- (f) the force between two electric charges is inversely proportional to the square of the distance between them (Coulomb’s law).

CHAPTER 2. UNITS

2.0 Terminology. In this text, “ln” will represent the Napierian (natural) logarithm, and “log” will represent the common logarithm.

In this chapter, the following symbols will be used to represent the numerics (nondimensional numbers) indicated: [Note N2.0]

$$c_a = 0.980665$$

$$c_m = 0.45359237$$

$$c_0 = 2.99792458 \times 10^{10}$$

$$c_p = 1.01325$$

The use of these and of the numeric 0.3048 ($= 12 \times 0.0254$) will permit statement of the exact ratios between units of the same physical quantity.

2.1 Coherent systems of units. As indicated in Chapter 1, the relations among physical quantities are usually described by proportionalities. Quantitative measurement of these quantities requires that these proportionalities be converted into equations. This conversion requires statements of the units in which each physical quantity must be measured in order to make the equation valid. Modern publications usually make these statements explicitly. However, earlier publications sometimes tacitly assumed the reader’s knowledge of the system of units used by the author. In order to allow full understanding of such publications, which are still authoritative and often unique, a brief review will be given of those systems of units that have had wide acceptance. More complete treatments exist (Refs. 2-1 to 2-3).

The systems of units were constructed by

(a) arbitrarily selecting a unit value for five fundamental physical quantities (four of these have invariably been mass, length, time, and temperature), and

(b) deciding on the form of the equations to be used to define all the remaining quantities in terms of the fundamental ones. This second step is tantamount to selecting the constants of proportionality that will convert relations like (1.1-1) into equations. In fact, the use of the defining equations in Tables 1-1 to 1-5 and the selection of the constant β that appears in Table 1-3 fulfill the requirements of this step. A system of units constructed in this manner has been termed “a coherent system of units.”

Sections 2.2 to 2.7 provide brief descriptions of the systems of units in common use since the creation of the International Committee on Weights and Measures (CIPM) in 1889. Table 2-1¹ lists, for each of several physical quantities, magnitudes in these systems of units that are equal to each other. Table 2-2¹ lists some relationships among the units that are useful in converting from one system to another and in the preparation and use of equations describing physical phenomena.

¹For ease of later reference, Tables 2-1 through 2-3 appear at the end of this text on pages 215 to 219.

2.2 Traditional systems of mechanical units. Historically, the fundamental mechanical units developed and used in the late nineteenth century on the European mainland were the centimeter (cm), gram (g), and second (s). The system of units based on these was termed the *cgs* system. The units of force and energy were the dyne and the erg, respectively.

In English-speaking countries, two systems became common. In one, the fundamental units for length, mass, and time were the foot (ft), pound (lbm), and second (s); the system of units was termed the *fps* system, and the unit of force was the poundal (pdl). In the other system, the fundamental units for length, mass, and time were the foot, slug, and second; the system of units was termed the *fpsg* gravitational system (*fpsg* system), and the unit of force was the pound force (lbf). Newton's law in the two English systems was

$$(2.2-1) \quad 1 \text{ pdl} = 1 \text{ lbm} \times 1 \text{ ft/s}^2$$

$$(2.2-2) \quad 1 \text{ lbf} = 1 \text{ slug} \times 1 \text{ ft/s}^2$$

The relation among units for the same physical quantity is

$$(2.2-3) \quad \frac{1 \text{ pdl}}{1 \text{ lbf}} = \frac{1 \text{ lbm}}{1 \text{ slug}} = 0.03048/c_a \approx 1/32.17405$$

In emulation of the dual use of the word "pound," a unit of force termed the *gram force* (gf) also became common; thereupon, the unit of mass became the *gram mass* (gm). The relations among the units of mass and force are

$$(2.2-4) \quad 1 \text{ lbf} = g_n \times 1 \text{ lbm}$$

$$(2.2-5) \quad 1 \text{ gf} = g_n \times 1 \text{ gm}$$

where g_n is the standard acceleration of gravity (980.665 cm/s^2 and $\approx 32.17405 \text{ ft/s}^2$). In publications where the unit of force is the pound force (or gram force) and the unit of mass is the pound mass (or gram mass), the factor g_n appears explicitly in some equations, often with just the symbol g .

2.3 Traditional systems of thermal units. The Celsius or centigrade degree ($^{\circ}\text{C}$) has been associated with the *cgs* system of mechanical units. Absolute temperature is then measured in kelvin (K) and energy in gram-calories (cal or gcal).

[In 1967, the CGPM elected to replace "degree kelvin" with "kelvin" (K)].

The Fahrenheit degree ($^{\circ}\text{F}$) has been associated with the *fps* and *fpsg* systems. Absolute temperature is then measured in degrees Rankine ($^{\circ}\text{R}$) and energy in British thermal units (Btu).

The relations among quantities in each system are

$$(2.3-1) \quad \text{K} = ^{\circ}\text{C} + \Theta_0$$

$$(2.3-2) \quad ^{\circ}\text{R} = ^{\circ}\text{F} + \Theta_0 - 32$$

Θ_0 = temperature (in K or $^{\circ}\text{R}$) of the triple point of H_2O

$$(2.3-3) \quad 1 \text{ gcal} = \text{energy to raise } 1 \text{ gm of } \text{H}_2\text{O} \text{ from } 15 \text{ to } 16 \text{ } ^{\circ}\text{C}$$

$$(2.3-4) \quad 1 \text{ Btu} = \text{energy to raise } 1 \text{ lbm of } \text{H}_2\text{O} \text{ from } 39 \text{ to } 40 \text{ } ^{\circ}\text{F}$$

2.4 Traditional systems of electric and magnetic units. In the field of electrostatics, the fifth fundamental unit has often been taken as the permittivity of a vacuum; the other four units are those of the *cgs* system.

The complete system is termed the cgs electrostatic system of units (*esu* system). Names of electrical quantities often bear the prefix “stat” (e.g., statampere, statfarad, stathenry, statvolt).

In this system, $\epsilon_0 = 1$ statfarad/cm, and $\mu_0 = (1/c_0^2)$ stathenry/cm.

In the field of magnetics, the fifth fundamental unit has often been taken as the permeability of a vacuum; the other four units are those of the cgs system. The complete system is termed the cgs electromagnetic system of units (*emu* system). Names of quantities often bear the prefix “ab” (e.g., abampere, abfarad, abhenry, abvolt). In this system, $\mu_0 = 1$ abhenry/cm and $\epsilon_0 = (1/c_0^2)$ abfarad/cm.

Some writers, like Gauss, have used $\epsilon_0 = 1$ statfarad/cm when dealing with electrostatic phenomena and $\mu_0 = 1$ abhenry/cm when dealing with magnetic phenomena. Consequently, some physical quantities are measured in esu, others in emu. The system is called the cgs Gaussian system.

2.5 Rationalization. The esu, emu, and Gaussian systems all use $\beta = 4\pi$ in the equations listed in Table 1-3. This factor enters principally because the surface of a sphere subtends 4π steradians at the sphere’s center. Some writers, like Heaviside and Lorentz, have set β equal to unity, thereby altering the relative magnitudes of some electrical and magnetic units, but reducing the frequency of occurrence of the numeric 4π . The use of $\beta = 1$ was termed “rationalization”; thereby, systems using $\beta = 4\pi$ were termed “unrationalized”. Heaviside and Lorentz also used the definitions of ϵ_0 and μ_0 that Gauss had used; the resultant system of units is termed the cgs Heaviside-Lorentz (*HL*) system.

2.6 Later systems of units. The “absolute” system of electric and magnetic units (which includes the ohm, ampere, volt, coulomb, farad, henry, watt, and joule) consists of simple decimal multiples of the emu that are of practically useful magnitudes. For about half a century prior to 1948, a set of “International” electric and magnetic units was in almost universal use. These differed very slightly from the absolute units because they were based on practically realizable international standards of the ohm and the ampere; units of other electric and magnetic quantities were defined in terms of these two standards. After 1948, improvements in technology made possible the practical realization of other absolute-system standards, and the older set of International units became obsolete. [Note N2.6]

In 1901, Giorgi pointed out that a simplification in the formulas for electromechanical phenomena could be achieved by a redefinition of the unit of permeability. The resultant unrationalized meter-kilogram-second system (*mks* system) uses the meter and kilogram as fundamental units, rather than the centimeter and gram, and takes $\mu_0 = 10^{-7}$ henry/meter as a fifth fundamental quantity. The system is, in effect, an mks electromagnetic system of units. Names of electric and magnetic units are those of the absolute system. The units of force and energy are the newton and joule, respectively.

The rationalized meter-kilogram-second system (*MKS* system) is like the mks system except that, since $\beta = 1$ instead of 4π in the formulas of Table 1-3, the value of μ_0 is taken as $4\pi \times 10^{-7}$ henry/meter. Units of force and energy remain the newton and joule, respectively.

2.7 The SI units. Gradual acceptance of Giorgi’s ideas has led to the adoption of the *Système International (SI)*—a rationalized system (Ref. 2-4). It is not to be confused with the “International” system discussed in Sec. 2.6. The current fundamental units are the meter, kilogram, second, kelvin, and ampere (rather than μ_0 , as in the MKS system). Additional basic units that have been defined are the mole as the unit of quantity and a psychophysical unit of luminous intensity, the candela. These are defined in Table 2-1.

Table 2-1 lists some common physical quantities, the names of the SI units in which they are measured, the abbreviations for these units, and the quantitative equals of these units in other systems of units. Reference 2-5 describes these units and other less common ones.

Table 2-3 lists the prefix symbols that designate multiples of a unit. (Mass is described as a multiple of a gram, not of a kilogram.) There is international agreement on the names of the units and on the symbols for the units, but, as indicated in Sec. 1.1, symbols for the physical quantity itself must be defined in each text.

Some dimensional constants are listed in Ref. 2-6.

2.8 Dimensionless units. Some physical variables are measured in units that are themselves dimensionless, usually because they represent the ratios between two quantities that have the same dimensions and that

are both measured in the same units. Alternatively, the units may represent some transcendental function of such a ratio. In particular, the logarithm of the ratio may be used if the range of values that the physical quantity may acquire in any single application is very large.

Example 2.8a. Plane angle represents the ratio of the length of a circular arc to its radius.

Example 2.8b. Solid angle represents the ratio of the area of a portion of a sphere's surface to the square of the sphere's radius.

Example 2.8c. Electric phase angle may represent the angle whose tangent is the ratio of the reactive component of a current to the resistive component of that current.

Example 2.8d. Phase angle between two time-varying sinusoidal signals of the same period may represent the ratio of the time displacement between the two signals (at some given ordinate, usually the mean value) to the period.

Example 2.8e. The *bel* (B) is used in the measurement of power P , but the most common unit is the *decibel* (dB). If M is the number of bels and m is the number of decibels,

$$(2.8-1a) \quad 0.1m = M = \log(P/P_0)$$

where P_0 is some other value of power. Thus, m and M are measures of the ratio between two quantities having the dimensions of power.

If dimensional considerations show that P is proportional to the square of the amplitude of some other physical variable X (like voltage, current, displacement, or pressure), then

$$(2.8-1b) \quad m = 20 \log(X/X_0)$$

where X_0 is some other value of X . Note that m remains a measure of power ratio.

Example 2.8f. The *neper* (Np) is also considered to be a measure of the ratio between two quantities that each have the dimensions of power P . It actually is a measure of the square root of the power ratio. If m_N is the number of nepers,

$$(2.8-2a) \quad m_N = (1/2) \ln(P/P_0).$$

In parallel with Eq. (2.8-1b), one may write

$$(2.8-2b) \quad m_N = \ln(X/X_0)$$

if P is proportional to X^2 . From Eqs. (2.8-1a) and (2.8-2a),

$$(2.8-3) \quad 20 m_N = m \ln 10$$

Example 2.8g. If, in Eq. (2.8-1), the quantity P_0 (or X_0) is assigned a fixed, standardized value termed a "reference value," the quantity P (or X) may be described as " m decibels re P_0 (or X_0)" or as being " m decibels above or below P_0 (or X_0)", depending on whether $P > P_0$ or $P < P_0$, respectively. The quantity m , then, is often said to be the "level" of whatever physical variable is being measured.

In acoustics, where the term "bel" originated, common choices are $P_0 = 1$ picowatt when P is sound power level, and $X_0 = 200$ picobar when X is sound pressure level. In the field of electrical transmission, $P_0 = 1$ milliwatt is sometimes used.

Example 2.8h. In psychoacoustics, *loudness level*, m_p , is measured in *phons*. The loudness level of a sound, in phons, is numerically equal to the value of m in Eq. (2.8-1b) when $X_0 = 200$ picobar, and X is the acoustic pressure of a 1 kHz tone that is judged equally loud by a standard auditor. Empirical curves exist that indicate the relation among m_p , X , and either the frequency f of pure tones or, alternatively, the center frequency f of relatively narrow bands of white noise (commonly, the bandwidth is one third or one ninth of the center frequency). (Ref. 2-7)

Loudness, L , of a complex sound is measured in *sones*. If m_s is the loudness of a sound in sones, then

$$(2.8-4) \quad 16 m_s = (p/p_0)^{\log 4}$$

where p_0 is 200 picobar and p is the acoustic pressure of a 1 kHz tone that is judged equally loud by a standard auditor. The perceived loudness is approximately proportional to the 0.6 power of the acoustic pressure.

Note that when $m_p = 40$ phon, we have $m_s = 1$ sone. In general,

$$(2.8-5) \quad \log m_s = (0.1 m_p - 4) \log 2.$$

Example 2.8i. The *noise figure*, N , of an electronic device has been represented by

$$(2.8-6) \quad N \text{ (in dB)} = 10 \log F$$

where F is the ratio between the actual noise power at the output of the device and the noise power that would have appeared at the output if the device itself had contributed no noise. (But see, also, Sec. 5-19.)

Example 2.8j. In audio communication, the *volume unit*, vu , of a complex audio waveform is defined as the numerical value of m in Eq. (2.8-1a) when P is the power developed in a 600-ohm resistance. Practical VU meters are ac voltmeters that measure the voltage across a 600-ohm load. The scales of such VU meters that conform to nationally accepted 1984 standards are marked to correspond to $P_0 = 2.5$ mW in 600 ohms.

Example 2.8k. In electrochemistry, pH is defined as

$$(2.8-7a) \quad pH = -\log a_{H^+}$$

$$(2.8-7b) \quad a_{H^+} \propto c_{H^+}$$

where a_{H^+} is hydrogen ion activity and c_{H^+} is hydrogen ion concentration. When pH is determined by measuring the emf e generated in an electrochemical cell,

$$(2.8-7c) \quad pH = 5040(e - e_0)T^{-1} \text{ kelvin per volt}$$

where e and e_0 are in volts, e_0 is an instrumental constant that may be dependent on T , and T is the temperature of the cell, in kelvin.

Example 2.8l. The *bolometric magnitude*, m_q , of a stellar object is given by

$$(2.8-8) \quad m_q = -2.5 \log (E_q/E_{q0})$$

where $E_{q0} = 2.5 \times 10^{-8}$ W/m² and E_q is the irradiance, in W/m², as measured by a bolometer.

The *absolute magnitude*, M_q , of a stellar object is defined as

$$(2.8-9) \quad M_q = m_q - 5 \log (l/l_0)$$

where $l_0 = 10$ parsec and l is the object's distance from the Earth, in parsec.

The *visual magnitude*, m_v , of a stellar object is given by

$$(2.8-10) \quad m_v = -2.5 \log (E_v/E_{v,0})$$

where $E_{v,0} = 2.6 \times 10^{-6}$ lux and E_v is the illuminance, in lux, as measured by a standard observer.

2.9 Dimensionless numbers. The dimensionless products that are commonly called "dimensionless numbers" were described in Sec. 1.3 and illustrated in Table 1-6. In algebraic form, they suffice for the purpose of dimensional analysis, as described in Sec. 1.5. However, before such a dimensional number can be used quantitatively in a numerical computation, its numerical value must be determined in such manner as to be independent of the particular unit in which each component physical quantity is measured; only then can diverse workers agree on a common "correct" value in any particular situation. The procedure for establishing the correct value of a dimensionless number with the fewest conversions is as follows:

Step 1. Each symbol representing a physical quantity is replaced by both the magnitude and the name of the unit in which that quantity has been measured.

Step 2. When the dimensionless product is formed, names of units are cancelled or otherwise treated as though they were algebraic quantities.

Step 3. If some names of units remain in the final product, they are caused to be cancelled by multiplying the product by unity, where unity is the ratio between appropriately selected equal amounts of the same physical quantity, chosen from Table 2-1 or Table 2-2.

Example 2.9a. Given $N_{Pr} = c\eta/k$, to find N_{Pr} for air when

$$c = 0.25 \text{ Btu} \cdot \text{lbm}^{-1} \cdot ^\circ\text{F}^{-1}, \quad \eta = 1.7 \times 10^{-4} \text{ P} \quad (\text{P} \equiv \text{poise}), \quad k = 0.014 \text{ Btu} \cdot \text{ft}^{-1} \cdot \text{hr}^{-1} \cdot ^\circ\text{F}^{-1};$$

$$\text{Step 1.} \quad N_{Pr} = 0.25 \frac{\text{Btu}}{\text{lbm} \cdot ^\circ\text{F}} \times 1.7 \times 10^{-4} \text{ P} \times \frac{\text{ft} \cdot \text{hr} \cdot ^\circ\text{F}}{0.014 \text{ Btu}}$$

$$\text{Step 2.} \quad N_{Pr} = \frac{0.25 \times 1.7 \times 10^{-4}}{0.014} \times \frac{\text{P} \cdot \text{ft} \cdot \text{hr}}{\text{lbm}}$$

$$\text{Step 3.} \quad \text{Multiply by } \frac{1 \text{ gm}}{1 \text{ P} \cdot \text{cm} \cdot \text{s}} \times \frac{1 \text{ lbm}}{454 \text{ gm}} \times \frac{30.48 \text{ cm}}{\text{ft}} \times \frac{3600 \text{ s}}{\text{hr}};$$

$$\text{then } N_{Pr} = 0.73.$$

Example 2.9b. Given $N_{Re} = vd/\nu$, to find N_{Re} when $\nu = 0.09 \text{ cm}^2/\text{s}$, $v = 400 \text{ ft/s}$, $d = 0.03 \text{ in}$:

$$\text{Step 1.} \quad N_{Re} = 400 \frac{\text{ft}}{\text{s}} \times 0.03 \text{ in} \times \frac{\text{s}}{0.09 \text{ cm}^2}$$

$$\text{Step 2.} \quad N_{Re} = \frac{400 \times 0.03}{0.09} \times \frac{\text{ft} \cdot \text{in}}{\text{cm}^2}$$

Step 3. Multiply by $\frac{12 \text{ in}}{\text{ft}} \times \frac{(2.54)^2 \text{ cm}^2}{\text{in}^2}$;

then, $N_{Re} = 1.03 \times 10^4$

Example 2.9c. Given $N_{Eu} = p/(\rho v^2)$, to find N_{Eu} when $p = 1 \text{ bar}$, $\rho = 1 \text{ g/cm}^3$, $v = 3 \text{ m/s}$: The combination of steps leads to

$$N_{Eu} = 1 \text{ bar} \times \frac{\text{cm}^3}{1 \text{ g}} \times \frac{\text{s}^2}{9 \text{ m}^2} \times \frac{10^5 \text{ N}}{\text{bar} \cdot \text{m}^2} \times \frac{1 \text{ kg} \cdot \text{m}}{\text{N} \cdot \text{s}^2} \times \frac{10^3 \text{ g}}{\text{kg}} \times \frac{10^{-6} \text{ m}^3}{\text{cm}^3} = \frac{100}{9}$$

2.10 Coherent equations. The relation among physical quantities is usually written as an equation representing the relation among algebraic symbols representing those quantities. Such an equation, rather than a proportionality, is necessary in order to establish the quantitative relation among those quantities. Such an equation is quantitatively correct if each algebraic symbol is replaced by its value in the same coherent system of units.

However, the equation remains correct, regardless of the unit in which any quantity is measured, if the following procedure is used: (1) Each algebraic symbol is replaced by both the magnitude and the name of the unit in which that quantity has been measured; (2) names of units are cancelled or otherwise treated as though they were algebraic quantities; (3) if the names of the units remaining are not to one's liking, the result is multiplied by unity as often as necessary, where unity is the ratio of appropriately selected equal amounts of the same physical quantity, like those listed in Tables 2-1 and 2-2.

The practice of replacing an algebraic symbol by both the magnitude and the name of the unit of measurement is a convenient expedient in those disciplines which have traditionally mixed units from several coherent systems. It also obviates the need for converting each unit of measurement into the unit of a single coherent system and usually reduces the number of arithmetic operations required. It is also convenient in using equations that involve a dimensional factor of proportionality, like those resulting from relation (1.1-3).

Example 2.10a. Use of coherent units. To find the force between two 1-coulomb charges one meter apart in vacuo:

Solution 1: In SI units (a rationalized system), $F = Q^2/(4\pi\epsilon_0 l^2)$, where $\epsilon_0 = 10^{11}/(4\pi c_0^2)$, $Q = 1$, $l = 1$. Then $F = c_0^2 \times 10^{-11}$ newton.

Solution 2: In mks units (an unrationalized system), $F = Q^2/(\epsilon_0 l^2)$, where $\epsilon_0 = 10^{11}/c_0^2$, $Q = 1$, $l = 1$. Then $F = c_0^2 \times 10^{-11}$ newton.

Solution 3: In cgs electrostatic units (an unrationalized system), $F = Q^2/(\epsilon_0 l^2)$, where $\epsilon_0 = 1$. Conversions: $Q = 1 \text{ C} \times (c_0/10) \text{ statcoulomb/C} = (c_0/10) \text{ esu of } Q$

$$l = 1 \text{ m} \times 100 \text{ cm/m} = 100 \text{ esu of } l$$

Then

$$F = \frac{c_0^2/100}{1 \times 10^4} \text{ dyne} = c_0^2 \times 10^{-6} \text{ dyne.}$$

Solution 4: In cgs electromagnetic units (an unrationalized system), $F = Q^2/(\epsilon_0 l^2)$ where $\epsilon_0 = (1/c_0^2)$.

Conversions: $Q = 1 \text{ C} \times (0.1 \text{ abcoulomb/C}) = 0.1 \text{ emu of } Q$

$$l = 1 \text{ m} \times 100 \text{ cm/m} = 100 \text{ emu of } l$$

Then

$$F = \frac{(0.1)^2}{(1/c_0^2) \times 10^4} \text{ dyne} = c_0^2 \times 10^{-6} \text{ dyne.}$$

Example 2.10b. Use of mixed units. Given $F = ma$, $m = 3 \text{ lb}$, $a = 2 \text{ ft/s}^2$, to express F in newtons:

$$F = \left(3 \text{ lb} \times \frac{2 \text{ ft}}{\text{s}^2} \right) \times \frac{c_m \text{ kg}}{\text{lb}} \times \frac{0.3048 \text{ m}}{\text{ft}} \times \frac{1 \text{ N} \cdot \text{s}^2}{1 \text{ kg} \cdot \text{m}} = 6 \times 0.3048 \times c_m \text{ newton}$$

Example 2.10c. Use of mixed units. Given $F = ma$, $m = 3 \text{ lbm}$, $a = 2 \text{ ft/s}^2$, to express F in lbf:

$$F = \left(3 \text{ lbm} \times \frac{2 \text{ ft}}{\text{s}^2} \right) \times \frac{1 \text{ lbf}}{g_n \text{ lbm}} = \frac{6}{32.2} \text{ lbf, since } g_n = 32.1740 \text{ ft/s}^2$$

Examples 2.9a, b, and c further illustrate the use of mixed units.

2.11 Numerical equations. The use of coherent equations of the type described in Sec. 2.10 is necessary in the mathematical analysis of physical situations. Their character has been taken for granted in most texts. However, where routine repetitive calculations are to be made, it is often more convenient to write an equation which is correct only when each quantity is measured in specified units. Quite often, in such cases, there is also the tacit assumption that other quantities remain invariant; the symbols for these other quantities usually do not even appear in the equation.

Example 2.11a. Equation (2.8-9) is often written as

$$(2.11-1a) \quad M_q = m_q - 5 \log l + 5$$

where l must be in parsecs, or as

$$(2.11-1b) \quad M_q = m_q + 5 \log p + 5$$

where p must be the heliocentric parallax in seconds of arc.

Example 2.11b. If v = speed of sound in air, a commonly used expression is

$$(2.11-2) \quad v = 49.0 \sqrt{T}$$

where air temperature T must be in $^{\circ}\text{R}$, v must be in ft/s, and the assumptions have been made that specific-heat ratio γ is 1.4 and that relative molar mass is 28.96.

Example 2.11c. The saturated vapor pressure of many chemical compounds has been written as

$$(2.11-3a) \quad \log p = B - 0.05223 A/T$$

where p must be in torr, absolute temperature T must be in kelvin, and A and B are tabulated constants for each compound.

The corresponding coherent equation is

$$(2.11-3b) \quad p/p_B = \exp [-\Delta W/(R_0 T)]$$

where p_B is a constant for each compound, ΔW is the compound's molar heat of evaporation, and R_0 is the universal gas constant. Thus,

$$(2.11-3c) \quad A = (\Delta W/R_0) \ln 10; \quad B = \log p_B.$$

Equation (2.11-3b) is valid for any units of pressure and molar heat, provided the procedure of Sec. 2.10 is followed.

Example 2.11d. The volumetric rate of flow \dot{V} of a gas through a very thin orifice of area A , caused by a difference Δp , under molecular-flow conditions ($N_{Kn} \gg 1$, where N_{Kn} = mean free path/orifice diameter), can be written as

$$(2.11-4a) \quad \dot{V} = 3.64A(\Delta p/p) \sqrt{(T/m_w)}$$

where \dot{V} must be in liter/s, A in cm^2 , and T in kelvin; m_w is the relative molar mass, and p is the upstream pressure in the same units as Δp .

The corresponding coherent equation is

$$(2.11-4b) \quad \dot{V} = A(\Delta p/p) \sqrt{[R_0 T / (2\pi m_w)]}$$

where R_0 is the universal gas constant.

Example 2.11e. If y is the deflection of the end of a spring under the weight of a mass attached to the end, if the spring itself is of negligible mass, and if f is the natural frequency of the mass-spring combination, a useful rule of thumb is

$$(2.11-5a) \quad f^2 y \approx 10$$

if f is in hertz and y is in inches. This equation also applies to a simple pendulum of length y .

The coherent equation that leads to Eq. (2.11-5a) is

$$(2.11-5b) \quad f^2 y = g / (4\pi^2)$$

where g is the local acceleration of gravity.

This subject is treated more completely in Sec. 8.4.

2.12 Conversion of numerical equations. A numerical equation that is correct only in one set of units can be converted to a numerical equation that is correct only in another set of units by the following procedure:

1. Replace each symbol x_i by x_i/u_i , where u_i is the name of the unit specified for the original equation.

2. Multiply each x_i/u_i by unity

$$1 = A_i u_i / (B_i v_i)$$

where v_i is the name of the new unit of x_i , and A_i and B_i are numerics obtained from Table 2-1.

3. After appropriate combinations of the numerics, the resultant equation will be correct for the new units.

Example: Given Eq. (2.11-2), to express T in kelvin (K) and v in km/hr:
Since step 1 leads to

$$v(\text{s/ft}) = 49.0 (T/^\circ\text{R})^{1/2},$$

we have

$$v \frac{\text{s}}{\text{ft}} \cdot \frac{1 \text{ hr}}{3600 \text{ s}} \cdot \frac{1000 \text{ ft}}{0.3048 \text{ km}} = 49.0 \left(\frac{T}{^\circ\text{R}} \cdot \frac{1.8 \text{ }^\circ\text{R}}{\text{K}} \right)^{1/2}$$

so that

$$v \text{ hr/km} = 72.1 \sqrt{(T/\text{K})}.$$

Hence

$$v = 72.1 \sqrt{T}$$

if v is in km/hr and T is in kelvin.

NOTES FOR CHAPTER 2

- N2.0** c_a is the numerical value of the standard acceleration of gravity when it is expressed in dekameters per (second)².
 c_m is the number of kilograms (mass) in one pound (mass).
 c_0 is the numerical value of the speed of light in vacuo when the speed is expressed in centimeters per second.
 c_p is the number of bars in one standard atmosphere.
 By international agreement, these numerics are exact.

N2.6. The differences between the International and the absolute units did not exceed 0.05 percent. For example,

- 1 International ohm = 1.00049 absolute ohm
- 1 International ampere = 0.99985 absolute ampere
- 1 International volt = 1.00034 absolute volt

For a few years after 1947, manufacturers of new precision instruments marked them explicitly to indicate that they were in conformity with the absolute system.

A systematic correction may be required to the marked values of precision resistors or other precision components manufactured before 1948 (hence, marked in the earlier International units) if they are to be used or compared with components manufactured after 1947, that are marked in absolute units, and if discrepancies on the order of 0.01 to 0.05 percent are significant.

CHAPTER 3. THE ACCURACY OF MEASUREMENTS

3.0 Introduction. Symbols. Among the many considerations that influence the choice of a measurement device or technique, one that can be described in precise mathematical language is the accuracy of measurement. Some other considerations that may be more important, yet less precisely definable, will be treated at the end of this chapter, when their relative weight may be better appreciated.

In making a measurement of a complex physical quantity by use of a number of instruments and operational steps, it is generally possible to state what inaccuracy was contributed by each instrument and by each operational step, and how each of these inaccuracies contributed to the inaccuracy of the final result. If this last value is lower than that required of the final result, the question arises whether appreciable reduction in time and in complexity of operations, and a concomitant increase in reliability, could have been achieved by using less accurate instruments and techniques; if the final inaccuracy is higher than that required of the final result, the problem is to determine at what points it is most feasible to improve accuracy in order to attain the goal of the measurement.

The ultimate accuracy of the final result is determined not so much by the accuracies of the instruments used as by the manner in which they are installed, the conditions under which they are used, and the properties or behavior of the physical variables that one is trying to measure. Each of these items is one of the independent variables upon which the final result depends. The understanding of the way these variables may behave, and of how they may contribute to the accuracy of the final measurement, is the principal goal of this chapter.

The symbols that are used frequently in this chapter are listed here. Infrequently used symbols will be defined as they are introduced.

e	error
E	fractional error
LE	limit of error
n	number of measurements
rms	root mean square
v	residual
w	weight
x	independent variable
y	dependent variable
Δy	systematic correction
δy	random error

Subscripts:

a	arithmetic
in	indicated
p	probable
tr	true
σ	root mean square
0	mean

Second subscripts:

0	of the mean
1	of a single observation

Section 3.19 contains some combination symbols that are used frequently in this text.

3.1 True and indicated values. In most experiments involving measurement of a physical quantity y , one is interested in the relation between the true value y_{tr} of the quantity and the measured or indicated value y_{in} .

Examples are

- (a) The voltage E_{tr} of a power supply whose output is indicated to be E_{in}
- (b) The resistance R_{tr} of a resistor whose marked value is R_{in}

In some applications, the distinction between y_{tr} and y_{in} may be considered negligible. Examples are

- (c) In kinematic calculations, the effects of latitude and longitude on the acceleration due to gravity are often considered negligible, and, for convenience, the standard acceleration g_n (Table 2-1) is assumed to exist at sea level
- (d) For convenience in making the buoyancy correction in weighing, air is often assumed to be at "standard temperature and pressure" (STP), namely, 0 °C and 1 standard atmosphere
- (e) For convenience in optical system design, the speed of light in air is often assumed to be c_0 cm/s (Sec. 2.0)
- (f) For convenience in electromagnetic calculations, the permeability and permittivity of air are often taken as μ_0 , and ϵ_0 , respectively (Table 2-1)

In other applications, the distinction between y_{tr} and y_{in} may be one of the principal objects or concerns of the experiment. Examples are

- (g) Determining the static temperature of a flowing gas from a thermometer immersed in the gas
- (h) Radiation pyrometry of a warm or incandescent object that is not a black body.

3.2 Sources of measurement error. Some sources of measurement error are of the following types:

- A. Variation in the physical quantity being measured
- B. Disturbance of the physical variable by the presence of the measuring instrument
- C. Errors introduced by the mode of installation of the instrument
- D. Deficiencies of the instrument itself
- E. Errors contributed by a human observer

Practical experience has shown that, in many applications, the above listing also is a listing in order of decreasing importance. Source A is often overlooked or neglected because it is not apparent or because it is difficult to establish quantitatively, especially if repetitive observations cannot be made. Sources B and C represent interactions between the instrument and the phenomenon being studied. Source D usually receives the greatest attention because a wealth of sophisticated laboratory equipment and techniques exists for its study; but this same technological competence also makes it possible for source D to contribute only slightly to the overall inaccuracy of the measurement. Source E is usually significant only when one approaches the limits of human perception in the measurement operation.

Examples of operations affected by type A sources are

- (a) Measuring the radiance of the Sun, or the radiance or albedo of the Earth
- (b) Measuring the intensity of radio signals reflected from the ionosphere
- (c) Measuring weak radioactivity in the presence of cosmic or terrestrial radiation
- (d) Measuring magnetic properties of iron alloys of uncertain history
- (e) Measuring a weak electrical signal in the presence of electronic noise
- (f) Measuring electrical conductivity of a chemically impure electrolyte
- (g) Measuring strains in the structure of a bridge carrying traffic
- (h) Measuring sound pressure levels in an outdoor environment
- (i) Measuring the drag of an airplane in turbulent air
- (j) Measuring fluid flow rate at the exit of a pump, turbine, or river

- (k) Deducing bulk velocity of a fluid in a pipe from a linear-velocity measurement along a single path, when the velocity distribution across the pipe is not known
- (l) Measuring bulk density or bulk velocity of a fluid when there is two-phase flow or nonisothermal flow because of the presence of local sources or sinks of heat
- (m) Measuring pressure in a high-vacuum chamber in which there are gas sources or sinks
- (n) Measuring local droplet size in a spray, by light scattering, when there are many droplets in the path of observation
- (o) Measuring local temperature or concentration of a hot gas by use of laser beams, when aerosols, particulates, or severe gas density gradients are in the optical path
- (p) Radiation pyrometry of a hot gas in which there is chemical activity or chemiluminescence
- (q) Radiation pyrometry of a turbine blade or wheel that reflects radiation from adjacent blades or other hot engine components.
- (r) Determining the rate of heat transfer between a fluid stream and an immersed thermometer bulb, in the presence of stream turbulence.

Examples of type B sources are

- (a) A voltmeter draws current from a circuit; an ammeter produces a voltage drop in a circuit.
- (b) A strain gage on thin sheet metal changes the local stiffness.
- (c) A vibration pickup on a small or flexible object changes the mode, amplitude, or frequency of the object's vibration.
- (d) An ionization gauge on a small high-vacuum chamber acts as a pump or as a source to change the pressure in the chamber.
- (e) A head-type, vortex-type, or turbine-type fluid flowmeter produces a pressure drop and a flowrate change in the pipe carrying the fluid.
- (f) A pitot-static probe or a temperature probe or the support of either affects the fluid flow in a pipe.
- (g) A thermocouple in contact with a small object changes the object's temperature by conduction of heat along the thermocouple leads.

Examples of type C sources are

- (a) Contact resistances affect the two-terminal measurement of an electrical resistance.
- (b) Capacitance of the cable connecting a piezoelectric pressure transducer to an electrometer affects the charge measurement.
- (c) Thermoelectric effects affect low-level dc emf measurements in a circuit with conductors of different materials.
- (d) Unclean windows affect radiation pyrometry or optical pyrometry of turbine blades or wheels in a jet engine (so does vignetting).
- (e) Dead time of a Geiger-Mueller tube limits the number of ionizing events that can be counted.
- (f) Improperly located current meters produce incorrect averaging of river flow.
- (g) Conduction of heat along the sheath affects the indication of a thermocouple probe measuring the temperature of fluid in a pipe or tank.
- (h) Heat loss by radiation and conduction affects the temperature acquired by a thermometer bulb in a stream of hot gas.
- (i) Proximity of the wall, or of adjacent probes, or of the support affects the indication of a static-pressure probe in a duct or wind tunnel.
- (j) The presence of a long connecting tube between pressure source and pressure pickup introduces errors in dynamic pressure measurement.

Type D sources (the instruments themselves) produce errors that are usually negligible in comparison with those of the other types. The indication of a sensor used to measure a physical quantity can usually be obtained with very little error; but the relation between that indication and the physical quantity that the sensor is intended to measure is affected by sources of types B and C.

Examples of operations affected by type E sources are

- (a) Matching of brightness by eye when using a disappearing-filament type of optical pyrometer or a line-reversal type of pyrometer
- (b) Matching luminous intensity in visual photometry or nephelometry
- (c) Establishing coincidence of image positions when using an optical range finder
- (d) Perception of indicating-accelerometer pointer position during a controlled acrobatic airplane maneuver
- (e) Matching or detecting sound amplitudes in audiometry.

3.3 Correction and error. The quantity that must be added algebraically to the observed indication y_{in} in order to obtain the true value y_{tr} is termed the *correction* Δy . The quantity $(-\Delta y)$ is termed the *error*.

$$(3.3-1) \quad \text{Correction} = \Delta y = y_{tr} - y_{in}$$

$$(3.3-2) \quad \text{Error} = e = -\Delta y = y_{in} - y_{tr}$$

3.4 Range and span. For many instruments, the following terms are used partially to define instrument characteristics.

Range: A statement of the lower limit y_{min} and the upper limit y_{max} of y that the instrument can measure

Span: The absolute value $|y_{max} - y_{min}|$ of the algebraic difference between the minimum and maximum values of the range

Full scale: The value of y_{max} . The term is of practical utility only when the lower limit of the range is zero or when the limits of the range are $\pm y_{max}$.

3.5 Methods of describing the accuracy of a single measurement. The relation between the true and the indicated values of a quantity y may be described in several ways. Some of these are

- (a) by stating the value of Δy
- (b) by stating the fractional value $\Delta y/y_{in}$ (also called the relative value) or its equivalent percentage value $100 \Delta y/y_{in}$. When this choice of designation is made, the fractional error

$$(3.5-1) \quad E = e/y_{in}$$

is used in place of Eq. (3.3-2) to describe the error.

- (c) by using a combination of (a) and (b); for example,

$$(3.5-2) \quad \text{Correction} = \Delta y + m\% \text{ of } y$$

- (d) by stating the correction as a fraction of span

$$\Delta y / |y_{max} - y_{min}|$$

- (e) by stating the correction as a fraction of full scale

$$\Delta y / y_{max}$$

- (f) by stating the ratio

$$y_{tr}/y_{in} = 1 + (\Delta y/y_{in})$$

The appropriate choice among these ways is usually deemed to be the one which is likely to have the same value over the entire range of the instrument. The nature of the instrument or of the measured quantity usually determines this choice.

3.6 Methods of describing the accuracy of an instrument. The accuracy of an instrument often can be described by providing a table or graph that gives the value of Δy , $\Delta y/y_{in}$, $\Delta y/\text{span}$, $\Delta y/y_{max}$, or y_{tr}/y_{in} over the entire range of the instrument (unless one of these values happens to be constant over the entire range). These values describing accuracy may have been obtained by instrument calibration or testing, or may have been estimated by analysis of the design and construction of the instrument.

However, one may prefer to characterize the instrument's accuracy by using a single compact expression that states the limit of inaccuracy that has been experienced in past use of the instrument or that can be expected in future use. Such choice is common among instrument manufacturers in preparing a catalog description.

The limit of inaccuracy is usually described in terms of the absolute value of Δy , $\Delta y/y_{in}$, $\Delta y/\text{span}$, or $\Delta y/y_{max}$. A common synonym for this (positive) quantity is "limit of error," and hence it will be designated by the abbreviation "LE." The selection of the form to be taken by the expression for limit of error is rarely arbitrary; just as indicated in Sec. 3.5, the form is determined by the nature of the physical variable being measured and by the circumstances under which the instrument is being used. Some simple forms are listed in the following section.

3.7 Limit of error. Some commonly used forms for describing the limit of error are

- LE = v , where v has the dimensions of y . This form is equivalent to $|\Delta y| < v$ and is often used to describe the accuracy of a resistor, capacitor, burette, or weight.
- LE = V percent of reading, where V is a numeric. This form is equivalent to $|\Delta y/y| < (V/100)$ and is often used to describe the accuracy of a power supply, spring, or clock.
- LE = V percent of span, where V is a numeric. This form is equivalent to $|\Delta y/(y_{max} - y_{min})| < (V/100)$ and is often used to describe the accuracy of a thermocouple-type pyrometer, a multiple-range signal analyzer, or a sound-pressure meter.
- LE = V percent of full scale, where V is a numeric. This form is equivalent to $|\Delta y/y_{max}| < (V/100)$ and is often used to describe the accuracy of an ammeter or of a pressure gauge.
- LE = $v + (V\%$ of reading), where v has the dimensions of y and V is a numeric. It is often written as

$$LE = v + V\%$$

This form is equivalent to $|\Delta y| < (v + yV/100)$ and is often used to describe the accuracy of a resistance decade box (where v reflects the resistance of internal connecting wires and switch contacts), a dc microvoltmeter or potentiometer (where v reflects the presence of thermal emf's), a dynamometer (where v reflects the presence of friction), or an interval timer (where v represents starting and stopping errors).

3.8 Nonlinear indications. The discussion of Secs. 3.4 through 3.7 has tacitly assumed that the instrument indication y is the same as the variable that is being measured, or is proportional to it. If the relation between the measured variable x and the indication y is nonlinear, the relation between Δx and Δy is obtained by appropriately differentiating the expression $y = y(x)$. The forms of representation of inaccuracy that were presented in Secs. 3.5 to 3.7 may still be used, but will prove convenient only in certain cases:

- If $y \propto x^n$, then, since

$$dy/y = n \cdot dx/x$$

the value of any percentage correction to y is n times the corresponding percentage correction to x . For example, in the relation between kinetic pressure p , fluid velocity v , and density ρ

$$p \propto \rho v^2, \quad (\text{Bernoulli's principle})$$

a 2-percent correction to a measured value of p implies a 1-percent correction to the deduced value of v .

(b) If $y \propto \log x$ as in Examples 2.8e, 2.8f, and 2.8l, then, since

$$dy = dx/x$$

a correction in the form of Δy becomes a correction in the form of $\Delta x/x$.

For example, if LE of $y = 0.03$, then LE of $x = 3\%$.

Thus, in this case, the most convenient form of representing the correction to y differs from the most convenient form of representing the correction to x .

3.9 Accuracy of an indirect measurement. If y is dependent on other variables x_1, x_2, x_3, \dots and is deduced from measurements of the x 's, then we may write

$$(3.9-1) \quad y = F(x_1, x_2, x_3, \dots)$$

and the relation between the correction to y and the corrections to the x 's is

$$(3.9-2) \quad \Delta y = \frac{\partial F}{\partial x_1} \Delta x_1 + \frac{\partial F}{\partial x_2} \Delta x_2 + \frac{\partial F}{\partial x_3} \Delta x_3 + \dots$$

In order to determine the derivatives, it is not necessary that Eq. (3.9-1) be known as an analytic, differentiable formula. Since Δy is merely a small correction, $\partial F/\partial x_i$ need be known only approximately. Thus,

- (a) If a tabulation of y exists for various values of the x_i , the value of $\partial F/\partial x_i$ may be approximated by $\Delta y_i/\Delta x_i$ in the vicinity of the values of x_1, x_2, x_3, \dots at which $\partial F/\partial x_i$ is desired. If the tabulation is sufficiently fine, Δy_i and Δx_i may be taken as the differences between adjacent tabular entries. If the tabulation is not sufficiently fine, the values of the increments may be deduced by quadratic interpolation among the tabular entries surrounding the point of interest. Sometimes, inspection of the entries will reveal that a retabulation of some function (e.g., the reciprocal, square, or logarithm) of the variables will facilitate interpolation.
- (b) An experiment may be performed in which x_i is changed by an amount Δx_i , while all other x 's are held unchanged, and the corresponding Δy_i observed [Note N3.9].

The various sources, A through E, listed in Sec. 3.2, usually represent separate variables. With this understanding, and if the Δ 's are, for the moment, assumed to represent inaccuracies in knowledge of the associated variable, then the relative magnitudes of the terms on the right side of Eq. (3.9-2) serve to indicate where any effort to improve accuracy (if such improvement is necessary) should be directed:

- (1) Terms of smaller magnitude represent operations or measurements that are already adequate.
- (2) Terms of larger magnitude represent operations or measurements where improved accuracy would be significant.

The implications of Eq. (3.9-2) are presented more fully in Sec. 3.28 as Eqs. (3.28-3) and (3.28-4).

3.10 Systematic and random errors. Errors of measurement may generally be characterized as either systematic or random.

Systematic errors are errors whose sign and magnitude are always the same under the same circumstances. If the sign and magnitude of a systematic error are known, the corresponding correction can be made to the measurement, so that the error is eliminated. If the nature of a systematic error is understood, but its magnitude is not known, deliberate steps may be taken to eliminate or to reduce its influence (Sec. 3.11). At the other extreme, the existence of a systematic error may not be suspected until a comparison with the results of another laboratory reveals a systematic difference in results, prompting a more intensive search for the systematic cause of the difference.

Random errors are errors of unknown sign, whose magnitude may only be estimated. If the random errors are of accidental origin, their magnitude may be estimated by making repeated observations under apparently identical circumstances.

The distinction between systematic and random errors is not unequivocal. If one does not choose to correct for a systematic error, it may in effect become a random one (see examples c through f in Sec. 3.1). If only the sign and LE of a systematic error are known, one may apply $(1/2)$ LE as a systematic correction and treat the remaining error as a random one whose limits are $\pm(1/2)$ of the original LE. Conversely, effective graphical and analytical techniques, like the method of least squares, are available that permit convenient partition of a measurement error into a systematic component and a random component. These techniques will be treated later; their selection and use involve a human choice among the convenience, complexity, and precision of describing the uncertainties. Furthermore, as one's experience and understanding increase, the sign and magnitude of an originally random error may be learned, so that it becomes a systematic, correctable one.

3.11 Systematic errors. Although an experiment may be attended by several systematic errors, only those errors need be considered whose magnitude may be sufficient to affect the result appreciably. Some expedients for dealing with systematic errors of uncertain but possibly significant magnitude are the following:

A. Appropriate experimental techniques may serve to cancel the error or to reduce it to an acceptably small magnitude.

B. If an experiment is intended to determine the effect upon y of one particular variable x_1 , where x_1 is only one of several independent variables, it may be possible to tolerate the existence of large systematic errors that do not depend on x_1 because these errors will remain constant throughout the experiment and can be canceled out of the final result. The experiment may be arranged deliberately to permit this situation by calling for the measurement only of dimensionless ratios. A considerable simplification of apparatus and technique and considerable improvements in speed, cost, and efficiency of operations are often realized by such a procedure.

Examples of systematic errors will be recognized among the examples listed in Sec. 3.2. Other examples and techniques for the reduction of errors are listed here:

Example 3.11a. The eccentricity of the divided drum of a micrometer screw produces a periodic, sinusoidal, error in the reading. As a correction, one may take the mean of the readings of two indexes 180° apart.

Example 3.11b. The eccentricity of an orifice plate in its holder produces an error in flow rate indication. However, (a) the same flow rate can be duplicated as often as desired without knowledge of the absolute value of flow rate, and (b) moderate changes in flow from some reference value (the *set point*) can be measured with negligible error. The ability to hold a set point and to correct for deviations from that set point, with only nominal knowledge of the exact flow rate, is a common requirement of many industrial processes.

Example 3.11c. Inequality in the length of the lever arms of a chemical balance results in a proportionate error in weighing. As a correction, the method of substitution eliminates the error.

Example 3.11d. Demagnetization of the permanent magnet in an electrobalance results in a proportionate error in weighing. As a correction, the method of substitution is applicable here. Furthermore, if the balance is accurately linear, the use of one accurate substitution weight will serve to establish the new proportionality constant. If the balance is slightly nonlinear, the use of two accurate substitution weights (one near full-scale value and the other near half-scale value) will also serve to establish the nonlinearity.

Example 3.11e. Self-heating of a resistance-thermometer bulb measuring the temperature of gas in a tank produces an error in temperature indication. However, repetition of the measurements, using 1/2 and 1/4 of the electrical power originally used, will permit

- (i) extrapolation to the indication that would be obtained if there were zero power dissipation in the bulb, and
- (ii) estimation of the optimum power dissipation that will provide the best compromise between self-heating error and sensitivity, to yield adequate accuracy with only a single measurement.

Example 3.11f. Background radiation from the walls of the enclosure housing the components of an infrared spectrometer, and from those components themselves, may cause an error in spectral amplitude measurement. If a light chopper is used to chop or modulate the beam entering the spectrometer, at a fixed frequency, and the detector is sensitive only to signals at and near to the chopping frequency, the effect of the background radiation may be eliminated.

Example 3.11g. Thermal expansion of an extensometer relative to the specimen produces a false indication of strain that is proportional to the temperature change. As a correction, the strain indication may be compared with that of a second extensometer attached to an unstrained specimen of the same material at the same temperature. Alternatively, the temperature may be monitored and a correction computed from the results of a separate experiment performed at zero strain and variable temperature.

3.12 Random errors. Random errors may arise from such sources as

- (a) inability to read to better than the least count of a digitized display
- (b) parallax in reading pointer position
- (c) friction and backlash in a mechanical device
- (d) vibration of the instrument support
- (e) contact potentials, in measuring small electrical voltages
- (f) electronic noise
- (g) changes in temperature and pressure when their effects are not understood
- (h) uncertainty in the magnitude of systematic corrections
- (i) fluctuations in the physical variable being measured, as illustrated by many of the examples of

type A sources listed in Sec. 3.2.

The "limit of error" (LE) is a useful, convenient means of describing the capability of a single instrument. However, when there are several independent variables in Eq. (3.9-1), the individual LE's may not be summed, because such summation would lead to an unrealistically pessimistic result whose occurrence has extremely low probability. A more realistic approach is to ask what is the most likely order of magnitude of the random error, or, more precisely, what is the probability that an error of given magnitude will not be exceeded. Answers to these questions use the language of statistics and require a study of repeated measurements of the physical variable of interest, or else the application of information obtained from previously performed repetitive measurements. The answers also depend on what intuitive assumptions are made concerning the nature of the random errors.

It is this statistical approach which is the principal concern of the remainder of this chapter. For a particular set of assumptions and a particular group of repetitive measurements, it will be possible to establish

- (a) the most probable value of the measured quantity,
- (b) measures of how well the individual observations already made agree with that most probable value, and
- (c) measures of how well future observations are likely to agree with that most probable value, under the assumption that future conditions of measurement are the same as those that existed during the past measurements.

The random error in y will often be written as δy ; the presence of the \pm sign in front of δy is implicit in its definition.

3.13 Most probable value of a quantity. Means. The median. If repetitive measurements of a quantity y are made in order to reduce the effect of random errors, one may then ask what is the most probable value of y that is to be deduced from the measurements. A similar question may be asked if one seeks a single value which will conveniently represent an average of several values that differ systematically or randomly. The answer depends on the nature of the quantity y and on the method and purpose of its measurement.

If $y_1, y_2, \dots, y_i, \dots, y_n$ are the n measurements of y and y_0 is the average or "mean" that is sought, some commonly used definitions of y_0 are the following:

A. The *arithmetic mean* y_0 is defined by

$$(3.13-1a) \quad n \bullet y_0 = \sum_1^n y_i$$

It is used when the individual measurements are considered to be of equal importance (i.e., when, *a priori*, the likely absolute error δy_i of a measurement is considered to be the same for all y_i).

The corollary *weighted arithmetic mean* y_0 is defined by

$$(3.13-1b) \quad y_0 \bullet \sum_1^n w_i = \sum_1^n w_i y_i$$

It is used when the individual measurements are considered to be of unequal importance and their relative importance is represented by their respective weights w_i . Some suggestion of how weights may be assigned will be given in Sec. 3.21.

B. The *geometric mean* y_0 is defined by

$$(3.13-2a) \quad y_0^n = y_1 y_2 \dots y_i \dots y_n = \prod_1^n y_i$$

All the y_i must be positive. This relation is tantamount to taking the arithmetic mean of the logarithm of the y 's.

$$(3.13-2b) \quad n \bullet \log y_0 = \sum_1^n \log y_i$$

The quantity y_0 in this expression has sometimes been called the *logarithmic mean*. The logarithm may have any base, including the Napierian and the common.

The geometric mean or the logarithmic mean is most appropriate when, *a priori*, the likely fractional error $\delta y_i / y_i$ of a measurement is considered to be the same for all y_i .

C. The *m*th root y_0 of the arithmetic average of the *m*th power of y is defined by

$$(3.13-3a) \quad n \bullet y_0^m = \sum_1^n y_i^m$$

A special case is the *harmonic mean* y_0 , defined by

$$(3.13-3b) \quad n/y_0 = \sum_1^n (1/y_i)$$

Another special case is the *root-mean-square* (rms) value y_0 defined by

$$(3.13-3c) \quad n \cdot y_0^2 = \sum_i^n y_i^2$$

D. In some cases, it may be appropriate to compute y_0 from the arithmetic mean of some other function of y . Thus, the arithmetic mean of the logarithm was taken in **B**.

E. The *median* of n measurements of y is that value y_0 such that half of the measurements are less than y_0 and half are more than y_0 (if n is even, y_0 lies halfway between two measured values). It is useful when the individual measurements are not considered to be of equal importance, but their relative importance is not known, *a priori*.

Just as one may take the arithmetic mean of powers, logarithms, or other functions of y in order to determine the most probable value y_0 , so may one, alternatively, take the median of powers, logarithms, or other functions of y .

All of the above formulas would yield the identical value of y_0 if all of the y_i were identical. Consequently, when all of the y_i are almost alike (i.e., when the δy_i or $\delta y_i/y_i$ are very small) one may choose, with insignificant loss of precision, a mean (or median) which is conveniently calculable or convenient for mathematical analysis, rather than the mean which is academically more appropriate.

Example 3.13a. Gas flows through a duct; gas pressure and linear velocity are constant across the cross section, but gas temperature varies with location in the cross section. In order to deduce mass flow rate from the equation

$$\text{mass flow rate} = \text{area} \times \text{linear velocity} \times \text{mean density,}$$

the harmonic mean of the absolute temperature is required.

Example 3.13b. Gas flows through a duct; gas pressure p and temperature T are constant across the cross section, but gas velocity, as deduced from the pressure differential Δp developed by a pitot-static tube, varies with location in the cross section. In order to deduce mass flow rate \dot{m} from the equation

$$\dot{m} = \int [2p \cdot \Delta p / (RT)]^{1/2} dA$$

where R is the specific gas constant and dA is the element of area, an area average of $\sqrt{(\Delta p)}$ is required.

Example 3.13c. To determine the average decibel level over an area, sound pressures are measured at several appropriate locations. The decibel level is then computed from the geometric mean of the pressures.

Example 3.13d. The following mean values exist in a high-vacuum system in which there is free-molecule flow (a high-vacuum enclosure is defined as one whose dimensions are much smaller than the mean free path of the gas molecules):

(i) The *mean free path* of the molecules is the arithmetic mean of the lengths of the paths of gas molecules between successive collisions with other molecules.

(ii) The *arithmetic mean speed* of the molecules. It determines the volumetric rate of flow of gas through a constriction like an orifice.

(iii) The *rms speed* of the molecules. It determines the pressure of the gas at any location; in particular, it determines the pressure on the walls bounding the system.

[Note N3.13]

3.14 Residuals. In any experimental measurement that is not perfectly accurate, the true value y_{tr} of the quantity of interest is not known; instead, the most probable value y_0 is deduced as the mean (or median) of a number of repetitive measurements of y , perhaps by processes like those illustrated in Sec. 3.13.

If a series of measurements y_i lead, by whatever process, to a most probable value y_0 , one defines the residual v_i as

$$(3.14-1) \quad v_i = y_i - y_0$$

The most probable value y_0 and the residual v can be calculated from the measurements y_i ; the error e cannot be obtained from Eq. (3.3-2) because y_{tr} is not known. Worthing (Ref. 3-1) has pointed out that, in most engineering measurements, the distinction between v and e is academic, because the difference between useful quantities derived from them is smaller than the uncertainty in knowledge of either quantity. However, the use of the residual is so firmly entrenched in the literature, that the distinction will be maintained in this text.

It is also sometimes convenient to deal with the fractional residual

$$(3.14-2) \quad V = (y_i - y_0)/y_i$$

3.15 Statistical descriptions of the past and the future. Past events can be described with all desired accuracy, simply by listing all the data. However, when there are many data, it may be possible to provide an adequate description by the shorthand of statistical terminology; such condensed description invariably implies that certain assumptions have been made about the nature of the data. The descriptions and the assumptions are approximations to the truth, and are justified by the convenience of a concise representation of the results and the convenience of any subsequent analysis of the results.

Future events cannot be predicted; they can only be estimated. The estimation is made by examining the statistical description of past events and assuming that the conditions that produced the past events will remain unchanged in the future, so that future performance can be an extrapolation of past performance.

Two questions of interest to those who use statistical terminology to describe a measurement are

1. How well are past data represented by their shorthand statistical description?
2. How well is the statistical description estimated to represent future results?

Answers to these questions depend on the assumptions made concerning the nature of the errors. These assumptions will be *a priori* assumptions because they must precede the data analysis and interpretation. In this text, only a few of the more common assumptions will be treated.

One of these is the nature of the random-error distribution. (Three common distributions will be described in Sec. 3.18.) Any assumed type of *distribution* will then be characterized quantitatively by the *dispersion* of the data that have been acquired. The name of the distribution merely identifies its shape; the dispersion describes numerical magnitudes that characterize that shape. The assumption of a distribution shape precedes the data analysis; the dispersion is a result of that analysis.

Another assumption, made *a priori*, is whether all data are likely to have the same uncertainty (error, δy), or to have the same fractional uncertainty (fractional error, $\delta y/y$), or whether each error is likely to have a weight w . The choice among these likelihoods is merely a choice of the form of the error, not of its numerical magnitude. It is made on the basis of prior experience and understanding of the nature of the physical quantity being measured and of the apparatus, techniques, and circumstances of the measurement operation. For example, an assumption that $\delta y/y$ is constant will place greatest weight on points near $y = 0$ and cannot be made if y may pass through zero. If the error is thought likely to be of the form $\delta y + Ey$ (Eq. (3.5-2)), with both δy and Ey of comparable magnitude, the use of w is required.

If a logarithmic scale of ordinates would be most appropriate to a graphical presentation of the data, then an *a priori* assumption that $\delta y/y = a$ constant is usually indicated; if a uniformly divided scale of ordinates would be most appropriate, then an *a priori* assumption that $\delta y = a$ constant is usually indicated.

3.16 The random-error distribution. Given a single instrument indication expected to represent the value of some physical variable, any one indication, after application of all known systematic corrections, may not represent the true value of the variable because of the presence of random errors. Because of such random

errors, it will never be possible to know the true value of the quantity. However, by repeated performance of the measurement operation, the totality of readings may be analyzed mathematically to establish the most probable value of the quantity; furthermore, this most probable value may be made to approach the true value by a sufficient increase in the number of repetitive observations. The extent to which the repetitious measurements need to be made depends on how accurately the quantity must be known; the extent to which the analysis can be carried depends on how well the law governing the distribution of errors is known.

A common assumption concerning random errors is that positive and negative errors are equally likely. This assumption will be made in the following text.¹

Consider that a very large number n of repeated measurements $y_1, y_2, \dots, y_i, \dots, y_n$ of the same quantity y_r possesses random errors whose absolute magnitudes are $e_1, e_2, \dots, e_i, \dots, e_n$ respectively. (Symbol e is thereby a shorthand symbol for $|e|$.) Let the largest possible value of these errors be e_m . The smallest possible value is zero. Divide the range of values of error into a number of equal intervals Δe , chosen to be small and yet to include a reasonably large number of errors in most of the intervals. Plot the number of errors that lie in each interval versus the value of the error at the midpoint of that interval. Such a plot is termed an *error-frequency distribution*.

As $n \rightarrow \infty$ to the extent that the concepts of the differential calculus may be used, a curve $f(e)$ may be faired through the points. As $\Delta e \rightarrow 0$ because $n \rightarrow \infty$, the probability dP that an error lies in the interval de is

$$(3.16-1) \quad dP = f(e) \cdot de$$

The probability that an error lies between the limits e_a and e_b is

$$(3.16-2a) \quad P = \int_{e_a}^{e_b} f(e) \cdot de$$

In particular, the probability that an error lies between 0 and e_m is a certainty, so that

$$(3.16-2b) \quad \int_0^{e_m} f(e) \cdot de = 1$$

The function

$$(3.16-3) \quad f(e) = dP/de$$

is termed a *probability coefficient* or a *frequency distribution function* or just an *error distribution*. Note that $f(e) \cdot de$ is dimensionless. Note also that, by Eq. (3.16-1), and the principles of the differential calculus, the probability of occurrence of any one error e is zero; only the occurrence of a band of errors, like $e_a \leq e \leq e_b$, as in Eq. (3.16-2a), has a nonzero probability.

3.17 Concise idealized measures of random-error dispersion. For the most common error distributions, three measures of dispersion have been widely accepted. These are the probable error, the average deviation, and the rms error (standard deviation). Since the assumption has been made that positive and negative errors are equally likely, and with symbol e representing $|e|$, we define, for $0 \leq e \leq e_m$ and $n \rightarrow \infty$,

¹If an error distribution is not symmetrical about the abscissa $e = 0$, it is said to be *skewed*. This asymmetry will be touched on briefly in the discussion of correlation coefficients (Sec. 4.12) but otherwise will not be treated.

$$(3.17-1) \quad A_0 = \int_0^{e_m} f(e) \cdot de$$

$$(3.17-2) \quad A_1 = \int_0^{e_m} e \cdot f(e) \cdot de$$

$$(3.17-3) \quad A_2 = \int_0^{e_m} e^2 \cdot f(e) \cdot de$$

We then have the following mathematical definitions of the three measures of dispersion of any form of error distribution.

The *probable error* (e_p or p or $p.e.$) is the abscissa which divides the curve $f(e)$ vs e into two equal areas.

$$(3.17-4) \quad \int_0^p f(e) \cdot de = A_0/2$$

The *average deviation* (e_a or a or $a.d.$) is the abscissa which is the center of moment of area of the curve $f(e)$ vs e .

$$(3.17-5) \quad e_a = A_1/A_0$$

The *rms error* (e_σ or σ or *standard deviation*) is the abscissa which is the center of the second moment of area of the curve $f(e)$ vs e . It is the positive square root of the *variance* which is given by

$$(3.17-6) \quad e_\sigma^2 = A_2/A_0$$

When the number of observations is so large that $f(e)$ may be treated as a continuous function for all practical purposes, alternative phrasings of these definitions are

The *probable error* e_p is such that half of all the errors (in absolute magnitude) are smaller than e_p and half of the errors are larger than e_p .

The *average deviation* e_a is the arithmetic mean of the absolute values of all the errors.

The *rms error* e_σ is the square root of the arithmetic mean of the squares of all the errors.²

3.18 Gaussian, uniform, and median error distributions. Some common random-error distributions will be identified here. It is assumed that all measurements have been corrected for known systematic errors.

I. The idealized *Gaussian distribution* is the result of the following assumptions:

- A. Positive and negative errors are equally likely.
- B. All magnitudes of error are possible.
- C. The frequency of occurrence of an error depends on its magnitude.
- D. Large errors are less likely than small ones.
- E. The likelihood of very large errors is very small.
- F. All measurements have been made with equal care.
- G. The number of measurements is large enough to warrant the application of the differential calculus.

²One of the uses of these measures of dispersion is to establish whether two values of a quantity are significantly different, when each value represents the mean of a number of observations. Each mean has its own measure of dispersion. If the ratio (> 1) between the two measures is less than 1.5 to 2, it is unlikely that the difference between the two values is significant. This matter is treated a little more precisely in Sec. 3.22.

Then if, *a priori*, each value of y is deemed likely to have the same random error δy ,

(a) the most probable value of y is the arithmetic mean y_0 (Eq. 3.13-1a);

(b) the error distribution $f(e)$ [Eq. 3.16-3] is shown in Fig. 3.18(a) and is given by the equation

$$(3.18-1) \quad f(e) = (1/\sigma)(1/\sqrt{2\pi}) \exp [-e^2/(2\sigma^2)];$$

(c) the idealized measures of dispersion (Sec. 3.17) are in the ratios

$$(3.18-2) \quad e_p : e_a : e_\sigma = 0.6745 : \sqrt{2/\pi} : 1$$

and

$$(3.18-3) \quad e_\sigma = \sigma.$$

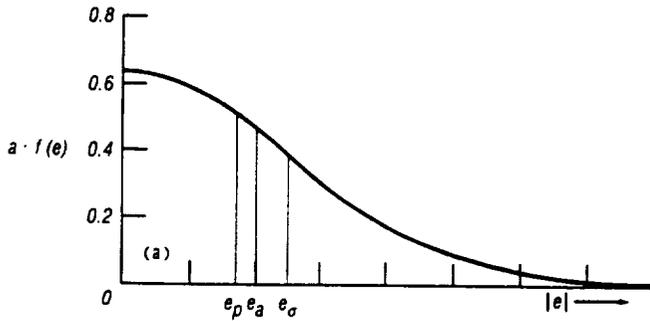
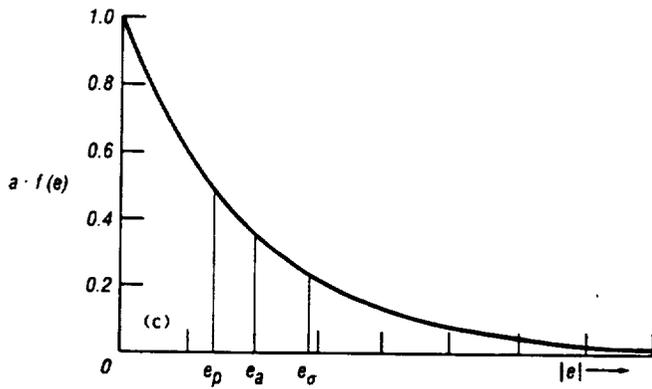
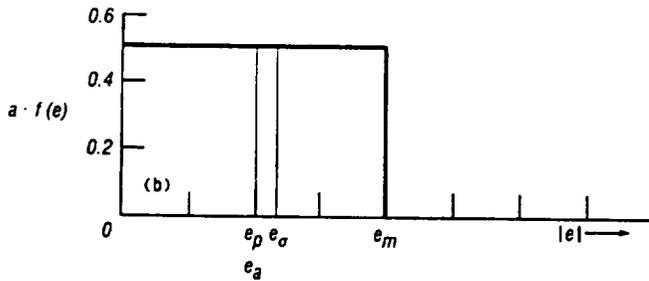


TABLE 3.18.1.—RELATIONS AMONG ERRORS OF THE GAUSSIAN DISTRIBUTION
Value of Y/X

Y	X				
	e_p	e_a	e_σ	e_{90}	e_{95}
e_p	1	0.845	0.674	0.410	0.344
e_a	1.183	1	$\sqrt{2/\pi}$	0.485	0.407
e_σ	1.483	$\sqrt{(\pi/2)}$	1	0.608	0.510
e_{90}	2.44	2.06	1.64	1	0.84
e_{95}	2.91	2.46	1.96	1.19	1



- (a) Gaussian.
- (b) Uniform.
- (c) Median.

Figure 3.18.—Error distributions.

TABLE 3.18.2.—ERRORS CORRESPONDING TO VARIOUS PROBABILITIES AND ODDS

[Errors are expressed as multiples of probable error p , average deviation a , or rms error σ .]

C*	P+	Gaussian distribution			t-distribution			Uniform distribution		Median distribution		
		e/p	e/a	e/σ	e/σ = e/[(Σv ²)/n] ^{1/2}			e/p	e/σ	e/p	e/a	e/σ
					n=16	n=8	n=4					
1:1	50.	1		0.67	0.69	0.71	0.77	1	0.87	1	0.69	0.49
1:1.4	42.		1									
1:2.1	32.			1								
1:4	20.			1.28	1.34	1.42	1.64	1.6	1.39	2.32	1.61	1.14
1:4.6	17.7	2										
1:5.7	15.			1.44	1.52	1.62	1.92	1.7	1.47	2.74	1.90	1.34
1:8	11.1		2									
1:9	10.			1.64	1.75	1.90	2.35	1.8	1.56	3.32	2.30	1.63
1:19	5.			1.96	2.13	2.37	3.18	1.9	1.65	4.32	3.00	2.12
1:21	4.55		2									
1:22	4.3	3										
1:32	3.			2.17	2.40	2.71	3.90	1.94	1.68	5.06	3.51	2.48
1:49	2.		3									
1:59	1.67			2.33	2.60	3.00	4.54	1.96	1.70	5.64	3.91	2.77
1:370	0.27		3									

*Odds that a greater error will occur
 +Probability of exceeding the tabulated error, percent

(See Footnote 3.)

The relations among the various errors are indicated in Table 3.18.1, which also includes errors e_{90} and e_{95} that are exceeded only 10 or 5 percent of the time, respectively. Table 3.18.2 lists the probability P that an error will exceed a given multiple of the common measures of dispersion. The percentage (100-P) has been termed a *confidence limit*, and has been used to indicate the reliability of an estimate of future results. The information on probabilities is restated in an alternate form, in Table 3.18.2, by giving the odds C of exceeding a given magnitude of error.

In practical engineering, the Gaussian distribution may be truncated because observations whose deviations from the mean exceeded some value like $3e_a$ or $3e_\sigma$ were rejected as grossly unreliable. (A common cause is a typographical error.) Such truncation has negligible effect on most of the ratios shown in Table 3.18.1 and in Eq. (3.18-2), although the magnitudes of the measures of dispersion (as in Eq. 3.18-3) are reduced by the rejection.

[Note N3.18]

II. The idealized *uniform distribution* is the result of the following assumptions:

- A. Positive and negative errors are equally likely.
- B. The absolute magnitude of an error cannot exceed e_m .
- C. For $|e| \leq e_m$, the frequency of occurrence of an error is independent of its magnitude.
- D. All errors are equally likely, as long as $|e| \leq e_m$.
- E. All measurements have been made with equal care.
- F. The number of measurements is large enough to warrant the application of the differential calculus.

³A fourth measure of dispersion, $h = 1/(\sigma \sqrt{2})$, called the *index of dispersion*, is used less frequently in present-day engineering applications of the theory of errors.

Then if, *a priori*, each value of y is deemed likely to have the same random error δy ,

(a) the most probable value of y is the arithmetic mean y_0 (Eq. 3.13-1a);

(b) the error distribution $f(e)$ [Eq. 3.16-3] is shown in Fig. 3.18(b) and is given by the equation

$$(3.18-4) \quad f(e) = 1/(2e_m) \quad |e| \leq e_m;$$

(c) the idealized measures of dispersion (Sec. 3.17) are in the ratios

$$(3.18-5) \quad e_p: e_a: e_\sigma = 1: 1: (2/\sqrt{3})$$

and

$$(3.18-6) \quad e_a = a = e_m/2$$

The probability P and the odds C of exceeding a given error are listed in Table 3.18.2.

III. The idealized *median distribution* is the result of the same assumptions as those for the Gaussian distribution, except that previously stated assumption *F* is replaced by

F. All measurements were not made with equal care and their relative reliability is not known in advance.

Then

(a) the most probable value of y is the median (Sec. 3.13.E.)

(b) the error distribution $f(e)$ [Eq. 3.16-3] is shown in Fig. 3.18(c) and is given by the equation

$$(3.18-7) \quad f(e) = (1/a) \exp(-|e|/a)$$

(c) the idealized measures of dispersion (Sec. 3.17) are in the ratios

$$(3.18-8) \quad e_p: e_a: e_\sigma = \ln 2: 1: \sqrt{2}$$

and

$$(3.18-9) \quad e_a = a$$

IV. In any of these distributions, if, *a priori*, each measurement of y is not deemed likely to have the same random error but is deemed likely to have a different random error δy_i , then y should be replaced by an appropriate function of y that may be presumed to have constant error. In particular,

(1) If y is deemed to have constant fractional error $\delta y/y$ then y should be replaced by its logarithm; y_0 becomes the logarithmic mean or median of the data (Sec. 3.13), and e is replaced by E , the nondimensional fractional error (Eq. 3.5-1).

(2) If y is deemed to have a weight w , then y should be replaced by wy , the quantity y_0 becomes the weighted mean (Eq. 3.13-1b) or weighted median, and e is replaced by $e\sqrt{w}$. (Weighted observations will be treated in Secs. 3.19 and 3.21).

3.19 Practical measures of random-error dispersion. The names and definitions of Sec. 3.17 are so convenient that they are used to describe the dispersion of data even when the number n of measurements is small and $f(e)$ does not have the analytic character implied in Eqs. (3.17-1 to -3). (Analytic character means that $f(e)$ is continuous and has a continuous derivative.) The integrals are replaced by summations, the errors are replaced by residuals, and the formulas are modified to allow for the fact that the most probable value y_0 derived from repetitive measurements y_i ($i = 1, 2, \dots, n$) is not the true value y_{tr} , which is never

known. The actual formula for y_0 depends on the assumptions made concerning the nature of the errors, but the following terms and symbols usually apply.

The measures of error fall into two groups. Terms in one group, whose descriptions contain the words "of a single observation," pertain to past events. The terms describe how well the individual data already taken agree with the most probable value y_0 of the measurements. The numerical value of these measures of dispersion is almost independent of the number n of measurements that have already been made.

Terms in the second group, whose descriptions contain the words "of the mean" or "of the median," are useful for future use. The terms describe how well y_0 is likely to represent the true value y_{tr} of the measured quantity. This likelihood will increase as the number n of past contributing measurements increases; the numerical value of these measures of dispersion will usually be found to vary inversely with \sqrt{n} . The terms in this second group provide estimates of the likelihood that y_0 will represent y_{tr} in the future, assuming that there will be no change in the circumstances governing the measurement.

Case A. Equally-weighted measurements. Given n observations y_i ($i = 1, 2, \dots, n$) such that, *a priori*, all y_i are deemed to have equal uncertainty δy , the following definitions hold as substitutes for the equations of Sec. 3.17. (All summations are from $i = 1$ to $i = n$.)

The most probable value of y is the arithmetic mean

$$(3.13-1a) \quad y_0 = (\Sigma y_i) / n.$$

If the residuals are defined as

$$(3.14-1) \quad v_i = y_i - y_0$$

then

$$(3.19-1) \quad e_{a1} = (\Sigma |v_i|) / [n(n-1)]^{1/2}$$

$$(3.19-2) \quad e_{a0} = e_a(y_0) = e_{a1} / \sqrt{n}$$

$$(3.19-3) \quad e_{\sigma 1}^2 = (\Sigma v_i^2) / (n-1)$$

$$(3.19-4) \quad e_{\sigma 0} = e_{\sigma}(y_0) = e_{\sigma 1} / \sqrt{n}$$

$$(3.19-5) \quad e_{p0} = e_p(y_0) = e_{p1} / \sqrt{n}$$

where

e_{a1} average deviation of a single observation

e_{a0} average deviation of the mean

$e_{\sigma 1}$ rms deviation of a single observation

$e_{\sigma 0}$ rms deviation of the mean

e_{p1} probable error of a single observation

e_{p0} probable error of the mean

The probable error of a single observation is, in theory, that value, e_{p1} , such that half of the residuals (in absolute magnitude) are less than e_{p1} and half are greater. In practice, the value of e_{p1} is deduced from e_{a1} or $e_{\sigma 1}$, by a formula that depends on the assumed distribution of errors. For example, for a Gaussian distribution, by Eq. (3.18-2),

$$(3.19-6) \quad e_{p1} = 0.67e_{\sigma 1} \approx 0.85e_{a1}$$

and, for the uniform distribution, by Eq. (3.18-5),

$$(3.19-7) \quad e_{p1} = e_{a1} = e_m/2 \approx 0.87e_{\sigma1}$$

The dimensions of the e_x 's, the quantities on the left side of Eqs. (3.19-1 to -5), are the same as the dimensions of y_i .

Differentiation of Eq. (3.19-3) shows that making Σv_i^2 a minimum leads to the definition of y_0 given by Eq. (3.13-1a). Conversely, the most probable value y_0 may be deduced by finding the condition that will minimize Σv_i^2 . This procedure of finding y_0 is thereby termed the *method of least squares*.

Case B. Measurements with constant fractional error. Given n observations y_i ($i = 1, 2, \dots, n$) such that, *a priori*, all y_i are deemed to have equal fractional uncertainty $\delta y/y$, the following definitions hold as substitutes for the equations of Sec. 3.17. (All summations and continuous products are from $i = 1$ to $i = n$.)

The most probable value of y is the geometric mean y_0 , which is also the antilog of the arithmetic mean of the logarithms of the y_i . If one defines the nondimensional residual

$$(3.19-8) \quad V_i = v_i/y_i = (y_i - y_0)/y_i$$

then the following nondimensional quantities (fractional errors) are useful.

$$(3.19-9) \quad E_{a1} = (\Sigma |V_i|)/[n(n-1)]^{1/2}$$

$$(3.19-10) \quad E_{a0} = E_a(y_0) = E_{a1}/\sqrt{n}$$

$$(3.19-11) \quad E_{\sigma1}^2 = (\Sigma V_i^2)/(n-1)$$

$$(3.19-12) \quad E_{\sigma0} = E_\sigma(y_0) = E_{\sigma1}/\sqrt{n}$$

$$(3.19-13) \quad E_{p0} = E_p(y_0) = E_{p1}/\sqrt{n}$$

where

- E_{a1} average fractional deviation of a single observation
- E_{a0} average fractional deviation of the mean
- $E_{\sigma1}$ rms fractional deviation of a single observation
- $E_{\sigma0}$ rms fractional deviation of the mean
- E_{p1} probable fractional deviation of a single observation
- E_{p0} probable fractional deviation of the mean

The probable fractional error of a single observation is, in theory, that value, E_{p1} , such that half of the values of $|V_i|$ are less than E_{p1} and half are greater. In practice, the value of E_{p1} is deduced from computed values of E_{a1} or $E_{\sigma1}$ by formulas that depend on the assumed distribution of errors. For example, for the Gaussian distribution, by Eq. (3.18-2),

$$(3.19-14) \quad E_{p1} = 0.67E_{\sigma1} \approx 0.85E_{a1}$$

and for the uniform distribution, by Eq. (3.18-5),

$$(3.19-15) \quad E_{p1} = E_{a1} = E_m/2 \approx 0.87E_{\sigma1}$$

Case C. Unequally-weighted measurements. Given n observations y_i ($i = 1, 2, \dots, n$) such that, *a priori*, each y_i is deemed to have a weight w_i , the following definitions hold as substitutes for the equations of Sec. 3.17. (All summations are from $i = 1$ to $i = n$.)

The most probable value of y is the weighted arithmetic mean y_0

$$(3.13-1b) \quad y_0 = [\Sigma(w_i y_i)] / \Sigma w_i.$$

If one defines the residuals

$$(3.14-1) \quad v_i = y_i - y_0$$

then the following definitions hold.

$$(3.19-16) \quad e_{a1} = [\Sigma(|v_i| \sqrt{w_i})] / [n(n-1)]^{1/2}$$

$$(3.19-17) \quad e_{a0} = e_a(y_0) = e_{a1} / \sqrt{\Sigma w_i}$$

$$(3.19-18) \quad e_{\sigma 1}^2 = [\Sigma(v_i^2 w_i)] / (n-1)$$

$$(3.19-19) \quad e_{\sigma 0} = e_\sigma(y_0) = e_{\sigma 1} / \sqrt{\Sigma w_i}$$

$$(3.19-20) \quad e_{p0} = e_p(y_0) = e_{p1} / \sqrt{\Sigma w_i}$$

where

- e_{a1} average deviation of a single observation of unit weight
- e_{a0} average deviation of the weighted mean
- $e_{\sigma 1}$ rms deviation of a single observation of unit weight
- $e_{\sigma 0}$ rms deviation of the weighted mean
- e_{p1} probable error of a single observation of unit weight
- e_{p0} probable error of the weighted mean

The probable error of a single observation of unit weight is, in theory, that value, e_{p1} , such that half of the products $|v_i| \sqrt{w_i}$ are less than e_{p1} and half are greater. In practice, e_{p1} is deduced from computed values of e_{a1} or $e_{\sigma 1}$ by formulas that depend on the assumed distribution of errors. For example, for the Gaussian distribution, by Eq. (3.18-2)

$$(3.19-21) \quad e_{p1} = 0.67 e_{\sigma 1} \approx 0.85 e_{a1}$$

and for the uniform distribution, by Eq. (3.18-5),

$$(3.19-22) \quad e_{p1} = e_{a1} = e_m / 2 \approx 0.87 e_{\sigma 1}$$

The important measures of unequally weighted measurements are e_{a0} , $e_{\sigma 0}$, and e_{p0} . The other measures, e_{a1} , $e_{\sigma 1}$, and e_{p1} are merely convenient intermediate steps in the computation of e_{a0} , $e_{\sigma 0}$, and e_{p0} .

It is noteworthy that assignment of a weight $w_i = 1/y_i^2$ is tantamount to solving the case of measurements with constant fractional error, so that the simpler treatment of Case B is usable. However, when the estimated error is of the form of Eq. (3.5-2), the treatment of Case C must be used.

For easier comparison, Table 3.19 lists the principal formulas of this section.

[Note N3.19]

TABLE 3.19.—SUMMARY OF PRACTICAL MEASURES OF DISPERSION

Line	Item	<i>A priori</i> assumptions		
		$\delta y = \text{constant}$	$\delta y/y = \text{constant}$	y_i has weight w_i
1	most probable value	$y_0 = (\Sigma y_i)/n$	$\ln y_0 = (\Sigma \ln y_i)/n$	$y_0 = \{\Sigma (w_i y_i)\}/\Sigma w_i$
2	residual	$v_i = y_i - y_0$	$V_i = 1 - (y_0/y_i)$	$v_i = y_i - y_0$
3	errors of individual observations	e_{p1} see line 9 or 10	E_{p1} see line 9 or 10	e_{p1} see line 9 or 10
4		$e_{a1} = (\Sigma v_i)/(n(n-1))^{1/2}$	$E_{a1} = (\Sigma V_i)/(n(n-1))^{1/2}$	$e_{a1} = [\Sigma (v_i /\sqrt{w_i})]/(n(n-1))^{1/2}$
5		$e_{\sigma 1}^2 = (\Sigma v_i^2)/(n-1)$	$E_{\sigma 1}^2 = (\Sigma V_i^2)/(n-1)$	$e_{\sigma 1}^2 = [\Sigma (v_i^2 w_i)]/(n-1)$
6	errors of the mean	$e_{p0} = e_{p1}/\sqrt{n}$	$E_{p0} = E_{p1}/\sqrt{n}$	$e_{p0} = e_{p1}/\sqrt{\Sigma w_i}$
7		$e_{a0} = e_{a1}/\sqrt{n}$	$E_{a0} = E_{a1}/\sqrt{n}$	$e_{a0} = e_{a1}/\sqrt{\Sigma w_i}$
8		$e_{\sigma 0} = e_{\sigma 1}/\sqrt{n}$	$E_{\sigma 0} = E_{\sigma 1}/\sqrt{n}$	$e_{\sigma 0} = e_{\sigma 1}/\sqrt{\Sigma w_i}$
9	Gaussian distribution	$e_{p1} = 0.67e_{\sigma 1} = 0.85e_{a1}$	$E_{p1} = 0.67E_{\sigma 1} = 0.85E_{a1}$	$e_{p1} = 0.67e_{\sigma 1} = 0.85e_{a1}$
10	Uniform distribution	$e_{p1} = 0.87e_{\sigma 1} = e_{a1} = e_m/2$	$E_{p1} = 0.87E_{\sigma 1} = E_{a1} = E_m/2$	$e_{p1} = 0.87e_{\sigma 1} = e_{a1} = e_m/2$

Some examples of the practical application of error distributions are

Example 3.19a. The probable error of the values in a five-place table of logarithms is 0.000 0025.

Example 3.19b. Mass production of electrical resistors yields a Gaussian distribution of resistance values, centered around the nominal or intended value. "Precision resistors" are obtained by selecting from the lot those resistors whose measured value falls within a limit of error (LE) of 0.1 percent. The error distribution among this group is substantially uniform, because the distribution represents the peak of the Gaussian curve. Another group, which has errors less than 1 percent, may be selected from the remainder, for sale at a lower price. The error distributions of this group will be Gaussian except for a trough between -0.1 and +0.1 percent. Presumably, for economic reasons, the e_{σ} of this group will be less than 0.5 percent so that there will be few discards. It is noteworthy that, where such manufacturing procedure is followed, all resistors have comparable stability; only the value of resistance may be in error (by less than 0.1 percent for the "precision" group; by not less than 0.1 percent for the remainder).

3.20 Percentiles An informative description of the dispersion of acquired data is the statement of the error magnitude which was not exceeded by most of the data. No particular error distribution need be implied. For example, one may give the 90th percentile (ninth decile) e_{90} or the 95th percentile (nineteenth vigintile) e_{95} , or one may simply state that " M percent of the deviations from the mean value y_0 were less than e_M ." Fractional errors may be treated similarly.

3.21 Weighting of observations. If a number of observations y_i is made of a quantity y and each observation has a random error, then, after all systematic corrections have been made, the most probable value of y is the weighted arithmetic mean of the observations, where the weight w_i to be assigned to any observation y_i is inversely proportional to the square of the random error of that observation. Equation (3.13-1b) may then be applied.

The assignment of weights requires only an assumption of the relative magnitudes of the random errors of the individual y_i . If there is no previous information available, then *a priori* assumptions are required, as indicated in Sec. 3.16. If the y_i represent values determined by previous repetitive measurements, so that each y_i already has an associated probable, average, or rms error, then the individual weight w_i is to be taken as the reciprocal of the square of that error.

Example 3.21a. If all measurements have the same uncertainty δy , then w_i may be taken as unity.

Example 3.21b. If all measurements have equal fractional error $\delta y/y$, then $\delta y = y \times$ a constant and w_i may be taken as $1/y_i^2$ in order to determine the arithmetic mean of y . However, all measurements have equal weight in determining the geometric mean or the logarithmic mean of y .

Example 3.21c. If $y = au^2$ and u has constant uncertainty δu , then

$$\delta y = 2au \cdot \delta u = 2 \cdot \delta u \cdot \sqrt{ay} = \sqrt{y} \times \text{a constant}$$

so that w_i may be taken as $1/y_i$ in order to determine the arithmetic mean of y .

Example 3.21d. If $y = au^2$ and u has constant fractional error $\delta u/u$, then

$$\delta y/y = 2 \cdot \delta u/u = \text{a constant}$$

so that

- (i) w_i may be taken as $1/y_i^2$ in determining the arithmetic mean of y
- (ii) w_i may be taken as unity in determining the geometric mean or the logarithmic mean of y .

Example 3.21e. The thermal conductivity k_0 , at STP, of H_2 gas taken from the same storage cylinder is determined by three independent laboratories. They respectively report the values 0.0167 ± 0.0003 , 0.0169 ± 0.0002 , 0.0168 ± 0.0004 W/(m•K), where the quantities after the \pm sign represent probable errors. Then the most probable value of k_0 is

$$0.0001[(167/9) + (169/4) + (168/16)]/[(1/9) + (1/4) + (1/16)] = 0.01683$$

with a probable error of 0.00006.

(A problem of this type is treated in greater depth in Sec. 3.32.)

3.22 Uncertainty in the practical measures of dispersion. The measures of dispersion listed in Sec. 3.19 are obtained by using a summation of residuals as an approximation to the integration of errors. These approximations themselves have uncertainties. Consequently, if e_x (or E_x) represents any one of the errors tabulated in Sec. 3.19, the more precise statement of e_x or E_x is

$$(3.22-1) \quad (e_x \text{ or } E_x) \cdot [1 \pm \beta / (n - 1)^\gamma]$$

where β and γ are numerics. Their magnitudes depend on the nature of the error distribution. The magnitude of β also depends on whether $\beta / (n - 1)^\gamma$ is intended to represent the probable, average, or rms fractional error of e_x or E_x ; these three magnitudes will be written as β_p , β_a , and β_σ , respectively.

Expression (3.22-1) shows that the validity (not the values) of the various e 's is diminished as the number n of observations is diminished. It also serves to establish whether the distinction between probable, average, or rms errors (e_p , e_a , or e_σ) is significant or merely academic, and to define the number of significant figures that are warranted in the statement of e_x or E_x .

For both the Gaussian and the median distributions, the fractional uncertainty in e_x or E_x is

$$(3.22-2) \quad \beta_x / \sqrt{(n - 1)}$$

where

$$(3.22-3) \quad \beta_p: \beta_a: \beta_\sigma = 0.67: \sqrt{(2/\pi)}: 1$$

However, for the Gaussian distribution

$$(3.22-4) \quad \beta_\sigma = 1/\sqrt{2}$$

and for the median distribution

$$(3.22-5) \quad \beta_\sigma = 1 .$$

The ratios in Eq. (3.22-3) are the same as those in Eq. (3.18-2) because the distribution of the “errors of the errors” is itself Gaussian for both the Gaussian and median error distributions. However, the uncertainty with which any measure of dispersion is known is larger for the median distribution than it is for the Gaussian distribution.

Example 3.22. Nine measurements are made of a quantity believed to follow a Gaussian error distribution.

- (i) if $e_\sigma \approx 0.1$, the representation of e_σ by two decimal digits is justified.
- (ii) if $e_\sigma \approx 0.8$, only a one-decimal-digit representation of e_σ is justified.

3.23 Rejection of observations. When a particular observation deviates from the mean by an amount that is several times the probable, average, or rms error, the decision on whether to reject it as spurious is a matter of human judgment. It does not follow from any mathematical laws. However, once the human judgment is made, it may be expressed in formal mathematical language; such formal expression is often useful if it is to be incorporated into a computer program or if the data are to be used by others less expert in understanding of the physical situation.

The opinions of others are useful in formulating a criterion for rejection of observations. Many such opinions have been published. They usually assume a Gaussian error distribution and usually express the criterion as a multiple of one of the standard measures of dispersion; often the multiple is itself a function of the number n of observations. Most choices lie between $3e_\sigma$ and $3e_\sigma$ of the data. Since the criterion is merely a formal mathematical description of one’s intuition, an understanding of the circumstances affecting the measurement—apparatus, technique, and observer—is just as important as are the numerical data.

After an observation has been rejected, the measures of dispersion must be recomputed. If the criterion for rejection is based on a measure of dispersion, it may be necessary to repeat the computation to preserve the formality. However, all such operations should be performed with full appreciation of the uncertainties indicated in Sec. 3.22.

3.24 Complete statement of the value of a physical quantity. A complete statement of the value of y requires a statement of the probable value y_0 and of the estimated inaccuracy using any of the forms indicated in Sec. 3.5. Some common forms of the complete statement are

- (a) $y_0 \pm e_x$ where e_x is the error of the mean, in units of y_0 ;
- (b) $y_0(1 \pm E_x)$ where E_x is a numeric representing the fractional error of the mean;
- (c) $y_0 \pm (e_x + 100E_x \%)$. Although not mathematically correct, this expression is accepted as a convenient representation of the more rigorous but more awkward expressions

$$(y_0 \pm e_x)(1 \pm E_x), \text{ or}$$

$$y_0 (1 \pm E_x) \pm e_x .$$

Traditionally, e_x and E_x represented probable errors in the English-language literature and represented rms errors on the European mainland. However, the use of rms errors has now become more common everywhere.

Some writers have preferred to place the limit of error (LE) after the \pm sign, usually defining the LE as the 95th percentile, although limits as high as 99.7 percent have been used. Because of these ambiguities, it is necessary that each writer explicitly state what type of error is represented by the entry after the \pm sign.

3.25 Truncation and rounding. The uniform distribution is significant when one deals with the expression of a quantity to m decimal places when (a) it is known that a more accurate expression would require more than m decimal places, and (b) the $(m + 1)$ th decimal is not known. Two types of situation which lead either to rounding or to truncation are illustrated by the following examples.

Example 3.25a. A four-place table of a transcendental function, like the logarithm, represents a *rounded* value. The tabulated value is the most probable value; its limit of error is 0.00005 and its probable error is 0.000025.

Example 3.25b. The indication of a commercial digital meter usually represents a *truncated* value. Assume that the indication is given to three decimal digits. Then the limit of error is almost -0.001 and there is a 50-percent probability that the error is less than -0.0005 . However, if the most probable value is taken to be 0.0005 more than the displayed value, then the assumptions of Sec. 3.18 apply, the limit of error becomes ± 0.0005 and the probable error becomes 0.00025.

Thus, a truncated digital display of a measured quantity does not represent the most probable value of that quantity, but the most probable value can be obtained by adding a systematic correction equal to one half of the least significant digit.

3.26 Average rate of random events, by counting. If a number n of discrete events, randomly separated in time, have all been counted in a time interval T (e.g., with a scaler), the computed frequency of occurrence

$$3.26-1) \quad \dot{n}_0 = n/T$$

is not necessarily the same as the frequency \dot{n}_{tr} that would have been computed if the counting time had been many times larger than T . However, \dot{n}_0 is the most probable value of the frequency, if it is the only information available. The rms fractional error $E_{\sigma_0}(\dot{n}_0)$ in knowledge of \dot{n}_0 is

$$(3.26-2) \quad E_{\sigma_0}(\dot{n}_0) = \delta\dot{n}_0/\dot{n}_0 = 1/\sqrt{n} \quad .$$

The expected distribution of readings is known as the Poisson distribution. This distribution is not a continuous function—it exists only for integer values of n . However, when n is moderately large, a smooth curve drawn through the ordinates is adequately approximated by a Gaussian distribution (Fig. 3.18(a) and Eq. (3.18-1)), in which σ is $1/\sqrt{n_0}$ and e is the individual error in \dot{n}_0 . (The curve through the ordinates of the Poisson distribution is slightly asymmetrical about $e = 0$ and is necessarily truncated on the left, since $n > 0$. When $n = 10$, the average difference between the two curves is less than 1 percent of the maximum amplitude (at $e = 0$).)

These idealized conclusions must be modified to correct for the imperfections of the instruments used in the measurements, limitations of the counting technique, and complications that may characterize the events being counted (D, C, and A sources of measurement error, respectively, in Sec. 3.2):

(a) **Dead-time error.** The idealized formula is predicated on the assumption that the counting instrument is capable of resolving successive events no matter how little separated they are in time. In practice, events counters have only a limited counting speed, characterized quantitatively by their *dead time* λ , an interval so short that two or more events occurring within the interval will be counted as only one event. If λ is small enough so that $\dot{n}_0\lambda \ll 1$, the most probable value of the total counts is no longer n , but rather

$$(3.26-3) \quad N = n(1 + \dot{n}_0\lambda)$$

where \dot{n}_0 is still given by Eq. (3.26-1). However, the most probable counting frequency now becomes

$$(3.26-4) \quad \dot{N}_0 = N/T$$

if T has been measured with negligible error. The rms fractional error in \dot{N}_0 remains

$$(3.26-5) \quad E_{\sigma 0}(\dot{N}_0) = \delta\dot{N}_0/\dot{N}_0 = 1/\sqrt{n}$$

(b) *Background error.* The results of this section find frequent application in the counting of radioactive disintegrations. In such cases, an additional error arises from the presence of spurious counts, due to the background of cosmic or ambient terrestrial radiation. This background must be measured separately. Every measurement must be corrected for the dead time of the events counter. If

N_B = corrected number of background events in time T

$$\dot{N}_B = N_B/T$$

N_T = corrected number of total counts

$$\dot{N}_T = N_T/T$$

then the most probable frequency of occurrence of the events of interest is

$$(3.26-6) \quad \dot{N}_0 = \dot{N}_T - \dot{N}_B$$

and the rms fractional error in \dot{N}_0 is

$$(3.26-7) \quad E_{\sigma 0}(\dot{N}_0) = [\sqrt{(N_T + N_B)}]/(N_T - N_B)$$

Equations (3.26-6) and (3.26-7) replace Eqs. (3.26-4) and (3.26-5).

3.27 Average rate of random events, by integration. If the rate of occurrence of randomly occurring events, as described in Sec. 3.26, has been determined by use of an integrating-type counting-rate meter (frequency meter, Note N3.27) rather than by use of a counter and timer, and if the ratemeter has a time constant τ , then

(a) if a single, instantaneous observation of the meter yields a frequency \dot{n} , the rms fractional error of \dot{n} is

$$(3.27-1) \quad E_{\sigma 1}(\dot{n}) = \delta\dot{n}/\dot{n} = 1/\sqrt{(2\dot{n}\tau)}$$

(b) if the meter is observed continuously over a time interval T and the arithmetic time average of its indication is \dot{n}_0 , the rms fractional error of \dot{n}_0 is

$$(3.27-2) \quad E_{\sigma 1}(\dot{n}_0) = (1 + \tau/T)^{-1} [1 + \tau/(2T)] / (\dot{n}_0 T)^{1/2}$$

(c) if m readings are taken of the ratemeter indication, at equally separated time intervals T/m , and the m readings have an arithmetic average \dot{n}_0 , the rms fractional error in \dot{n}_0 is

$$(3.27-3a) \quad E_{\sigma 0}(\dot{n}_0) = (A/B) / \sqrt{(2\dot{n}_0\tau)}$$

where

$$(3.27-3b) \quad A = [1 + (m - 1)(1 - r^2)]^{1/2}$$

$$(3.27-3c) \quad B = 1 + (m - 1)(1 - r)$$

$$(3.27-3d) \quad r = \exp[-T/(m\tau)]$$

Equations (3.27-2) and (3.27-3) are based on the assumption that the ratemeter had already been connected for a sufficient time to ensure the subsidence of starting transients; a criterion for this condition is that

- (1) the ratemeter indication no longer shows any appreciable steady, monotonic drift, or
- (2) the ratemeter had been connected for a time at least as great as

$$(3.27-4a) \quad \Delta t = \tau \ln \sqrt{2\dot{n}\tau}$$

or

$$(3.27-4b) \quad \Delta t = 4\tau,$$

whichever is smaller.

The corrections for background and for dead time of the counting chamber (like a Geiger-Mueller tube), given in Sec. 3.26, still apply.

3.28 Combination of errors. Single measurement of a function of several variables. Assume that a quantity z is computed, or otherwise determined, by measurement of k variables $u_1, u_2, \dots, u_j, \dots, u_k$ through the relationship

$$(3.28-1) \quad z = z(u_1, u_2, \dots, u_j, \dots, u_k)$$

where each variable u_j has a most probable value u_{j0} , a systematic error Δu_j , and a random error δu_j .

Then, prior to application of systematic corrections, the most probable value of z is

$$(3.28-2) \quad z_0 = z(u_{10}, u_{20}, \dots, u_{j0}, \dots, u_{k0}).$$

The systematic error in z_0 is, by Eq. (3.9-2)

$$(3.28-3) \quad \Delta z_0 = \sum (\partial z / \partial u_j) \cdot \Delta u_j$$

(All summations in this section are from $j = 1$ to $j = k$.) Because it is systematic, this quantity may be applied as a correction.

If, *a priori*, the random deviations from the mean are equally likely to be positive or negative, and if the variables u_j are independent of each other, then

$$(3.28-4) \quad (\delta z_0)^2 = \sum (\partial z / \partial u_j)^2 \cdot (\delta u_j)^2$$

(Two variables are said to be independent of each other if the value of each does not depend on the value of the other.)

Equation (3.28-4) differs from Eq. (3.28-3) and its equivalent, Eq. (3.9-2), in two important respects:

- (i) In Eqs. (3.28-3) and (3.9-2), the variables, u_j , need not be independent; in Eq. (3.28-4) the variables u_j must be independent—no one may depend on another, nor may any two depend on some common parameter.
- (ii) Because of the ambiguity in the sign of random errors, the summation of Eq. (3.28-4) involves squares of terms; random error terms are always additive—there can be no cancellation of terms.

As indicated in Sec. 3.9, each of the error sources, A through E, listed in Sec. 3.2, usually represents an independent variable. The relative magnitudes of the terms in the summation of Eq. (3.28-4) indicate the relative importance of the various error sources in their effect on the accuracy of z .

In Eq. (3.28-4), the errors may be of any type, provided they are all alike; the equation holds for either the probable, average, or rms errors of the means. If δu_j represents the probable error of u_{j0} , then δz_0 represents the probable error of z_0 .

Sometimes, the fractional error is more convenient and meaningful than the absolute error. The practical forms which Eqs. (3.28-3) and (3.28-4) take in specific instances are shown by the following examples. (Subscript zero has been omitted for brevity.)

Example 3.28a. If $z = u_1 + u_2 + u_3 + \dots$, then

$$\begin{aligned}\Delta z &= \Delta u_1 + \Delta u_2 + \Delta u_3 + \dots \\ (\delta z)^2 &= (\delta u_1)^2 + (\delta u_2)^2 + (\delta u_3)^2 + \dots\end{aligned}$$

Example 3.28b. If $z = u_1 u_2 u_3 \dots$, then

$$\begin{aligned}\Delta z/z &= (\Delta u_1/u_1) + (\Delta u_2/u_2) + (\Delta u_3/u_3) + \dots \\ (\delta z/z)^2 &= (\delta u_1/u_1)^2 + (\delta u_2/u_2)^2 + (\delta u_3/u_3)^2 + \dots\end{aligned}$$

Example 3.28c. If $z = u_1^{a_1} u_2^{a_2} u_3^{a_3} \dots$, then

$$\begin{aligned}\Delta z/z &= a_1 (\Delta u_1/u_1) + a_2 (\Delta u_2/u_2) + a_3 (\Delta u_3/u_3) + \dots \\ (\delta z/z)^2 &= a_1^2 (\delta u_1/u_1)^2 + a_2^2 (\delta u_2/u_2)^2 + a_3^2 (\delta u_3/u_3)^2 + \dots\end{aligned}$$

Example 3.28d. If $z = a_1 \ln u_1 + a_2 \ln u_2 + a_3 \ln u_3 + \dots$, then

$$\begin{aligned}\Delta z &= a_1 (\Delta u_1/u_1) + a_2 (\Delta u_2/u_2) + a_3 (\Delta u_3/u_3) + \dots \\ (\delta z)^2 &= a_1^2 (\delta u_1/u_1)^2 + a_2^2 (\delta u_2/u_2)^2 + a_3^2 (\delta u_3/u_3)^2 + \dots\end{aligned}$$

Example 3.28e. Power P generated in a resistor R is computed from m measurements of current I . Then the most probable value of P is $[(\Sigma I)/m]^2 R$. It is not $[(\Sigma I^2)/m]R$.

Example 3.28f. Power P generated in a resistor is determined by first making m measurements of current I and then making n measurements of voltage drop E . Then the most probable value of P is $[(\Sigma I)/m] \cdot [(\Sigma E)/n]$.

Example 3.28g. Power P generated in a resistor is determined by making m pairs of simultaneous measurements of current I and voltage drop E . Then the most probable value of P is $[\Sigma (EI)]/m$.

Example 3.28h. The relation between Nusselt, Prandtl, and Reynolds numbers (N , P , and R , respectively) for heat transfer between the central section of a long cylinder and a fluid flowing in a direction normal to the cylinder's axis is given by

$$N = 0.95 P^{1/3} R^{(0.31 + 0.037 \log R)}$$

if $0.1 \leq R \leq 200\,000$. (The formula is derived from Fig. 10-7 in Ref. 3-2.) When $R = 10^5$, 10-percent uncertainties in P and in R yield an 8-percent uncertainty in N .

Example 3.28i. A total radiation pyrometer measures the flux Φ_e emitted from a body, whose temperature is T and whose total emittance is e , in accordance with the formula

$$\Phi_e = B\sigma e(T^4 - T_0^4)$$

where T_0 is the pyrometer's temperature, σ is the Stefan-Boltzmann constant, and B is a calibration constant characterizing the geometric optics of the instrument and the installation. If $T_0 \ll T$, a 1-percent uncertainty in T is produced by a 4-percent uncertainty in e .

On the other hand, a monochromatic radiation pyrometer, which measures the radiant flux $\Phi_{e,\lambda} \cdot \Delta\lambda$ that is emitted from a body in a narrow wavelength interval $\Delta\lambda$ centered around a wavelength λ , obeys the equation

$$\Phi_{e,\lambda} \cdot \Delta\lambda \approx B e_\lambda \lambda^{-5} \exp[-c_2/(\lambda T)]$$

where $c_2 = 0.0144 \text{ K}\cdot\text{m}$ is Planck's second radiation constant and e_λ is the monochromatic emittance of the body. Then, a 1-percent uncertainty in T is produced by

- (i) a 5-percent uncertainty in e_λ when $\lambda T = 0.003 \text{ K}\cdot\text{m}$ (the product of λ and T that produces maximum spectral radiance)
- (ii) a 14-percent uncertainty in e_λ when $\lambda T = 0.001 \text{ K}\cdot\text{m}$. Thus, operation at a shorter wavelength is advantageous, if there is still adequate radiometric sensitivity.

Example 3.28j. In the range $0 \leq t \leq 40 \text{ }^\circ\text{C}$, the effect of temperature t on the emf E of a saturated Weston standard cell is given by

$$E = 1\,018\,220 - \Delta E - 53.97x - 0.71x^2 + 0.010x^3$$

where $x = t - 28 \text{ }^\circ\text{C}$, t is in $^\circ\text{C}$, E and ΔE are in microvolts, and ΔE is a systematic correction furnished by the manufacturer for each cell.

In the same temperature range, the effect of temperature on the emf of an unsaturated Weston standard cell is given by

$$E = (1\,019\,300 + \Delta E)(1 - 0.00283y - 0.112y^2 + 10.5y^3)$$

where $y = (t - 20 \text{ }^\circ\text{C})/1000$. (ΔE , a systematic correction for the particular cell, is so small that it may be neglected in computing the temperature correction.)

Then, the emf E will be known to 0.01 percent if $\delta(\Delta E) \ll 100$ and if

- (i) the saturated cell temperature is known to $2 \text{ }^\circ\text{C}$ when the cell's temperature is near $28 \text{ }^\circ\text{C}$ (the usual operating temperature);
- (ii) the unsaturated cell temperature is known to lie between 0 and $40 \text{ }^\circ\text{C}$.

Formulas (3.28-3) and (3.28-4) are not complete in a region where z is a maximum or minimum with respect to one of the variables u_j . The formulas fail completely if z is a function of only one variable u and $dz/du = 0$ at the point where errors are to be computed. Under such circumstances, the quantity

$$(\partial z / \partial u_j) \cdot \Delta u_j$$

must be replaced by

$$(\partial z^n / \partial u_j^n) \cdot (\Delta u_j)^n / n!$$

where n is the smallest positive integer for which the corresponding derivative is not zero.

Example 3.28k. In a particular electrometric titration wherein the potential difference u between two electrodes is related to the amount z of electrolyte added, the relation between the two variables in the vicinity of the equivalence point (a,b) is found to be given by

$$z - b = c(u - a)^3.$$

Then, at $u = a$,

$$dz/du = 0; \quad d^2z/du^2 = 0; \quad d^3z/du^3 = 6c$$

so that

$$\Delta z = c(\Delta u)^3; \quad (\delta z)^2 = c^2(\delta u)^6.$$

Example 3.28l. The relation between area A of a variable area channel and the Mach number M at that area, for a gas of constant specific-heat ratio k , is given by

$$A = (A_1/M)[(1 + bM^2)/(b + 1)]^{(b+1)/(2b)}$$

where $b = (k - 1)/2$ and A_1 is the area at which $M = 1$. To determine how the Mach number might change when the downstream channel area expands from A_1 to $A_1 + \Delta A$, we note that, at $M = 1$,

$$\partial A / \partial M = 0$$

$$\partial^2 A / \partial M^2 = (7k - 9)/(k^2 - 1) \equiv G$$

so that

$$(\Delta M)^2 = (2/G)(\Delta A/A_1).$$

Example 3.28m. The relative resistivity ρ/ρ_1 of manganin wire used for precision resistors, in the range $15 \leq t \leq 50$ °C, is given by

$$\rho/\rho_1 = 1 + 1.335 \times 10^{-4}x - 4.245 \times 10^{-5}x^2$$

where $x = t - 20$ °C, t is temperature in °C, and ρ_1 is the resistivity at 20 °C (0.00484 Ω•m).

The relative resistivity of manganin sheet used for ammeter shunts, designed to operate at about 45 °C, is given, for the range $15 \leq t \leq 80$ °C, by

$$\rho/\rho_2 = 1 - 3.28 \times 10^{-5}y - 2.048 \times 10^{-5}y^2$$

where $y = t - 45$ °C, and ρ_2 is the resistivity at 45 °C (0.00388 Ω•m). Consequently,

- (i) temperature must be known to within 1.5 °C if the value of a precision resistor is to be known to 0.01 percent;
- (ii) temperature must be known to within 7 °C if the value of a shunt is to be known to 0.1 percent.

3.29 The central limit theorem. It has been found that, when errors from various sources compound as given in Eq. (3.28–4), and each source has a different error distribution (although one that is symmetrical about $e = 0$), the error distribution of the result, z , closely resembles the Gaussian. The fact that the combination of errors from several independent sources tends to have a Gaussian distribution has been termed the “central limit theorem.”

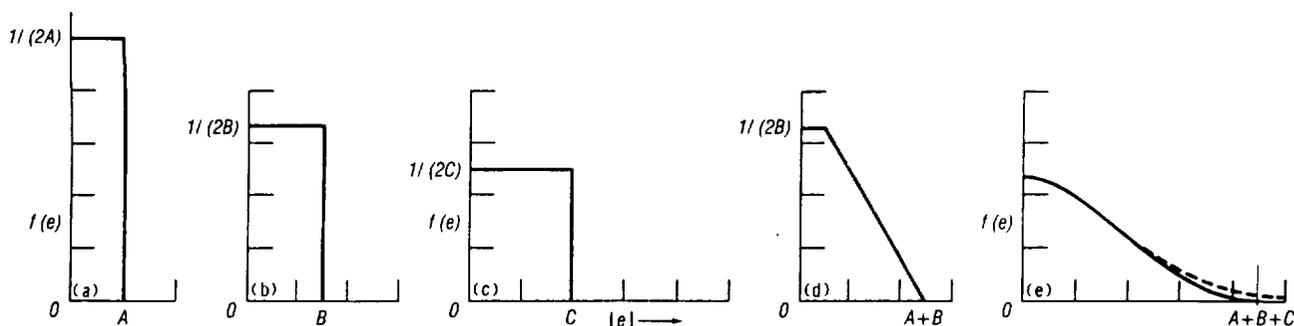


Figure 3.29.—Sums of uniform error distributions, to illustrate the central limit theorem.

This theorem can be illustrated by the combination of errors from several sources, each of which has a uniform distribution. Suppose that u_1 has a uniform error distribution over the range $-A \leq e_1 \leq A$, where $e_1 \equiv \delta u_1$, and that u_2 has a uniform error distribution over the range $-B \leq e_2 \leq B$, where $e_2 \equiv \delta u_2$, and $B \geq A$. These distributions are shown in Fig. 3.29(a) and (b). (The ordinates of all distributions have been chosen so that they all enclose the same area, corresponding to unit probability.) Then the error distribution of $z = u_1 + u_2$ has the form shown in Fig. 3.29(d). If, in addition, u_3 has a uniform distribution over the range $-C \leq e_3 \leq C$, (Fig. 3.29(c)) where $e_3 \equiv \delta u_3$, then the error distribution of $z = u_1 + u_2 + u_3$ has the form shown in Fig. 3.29(e)—a form hardly distinguishable from the Gaussian curve with the same average deviation or rms error, which is shown by the dashed line [Note N3.29].

3.30 Several measurements of a function of one or more variables. If the measurement of z described in Sec. 3.28 is repeated on several different occasions, leading to mean values $z_{01}, z_{02}, \dots, z_{0i}, \dots, z_{0m}$, each with its respective rms, average, or probable error $\delta z_{01}, \delta z_{02}, \dots, \delta z_{0i}, \dots, \delta z_{0m}$, then the most probable value of z is its Grand Mean z_{00} , which is the weighted average of the individual mean values, the weighting factor for each being the reciprocal of the square of the respective error (All summations are from $i = 1$ to $i = m$)

$$(3.30-1) \quad z_{00} = \left\{ \frac{\sum [z_{0i} / (\delta z_{0i})^2]}{\sum [1 / (\delta z_{0i})^2]} \right\}.$$

By Eqs. (3.19-18) and (3.19-19), the rms error $e_{\sigma 0}(z_{00})$ of the Grand Mean is given by

$$(3.30-2) \quad e_{\sigma 0}^2(z_{00}) = e_{\sigma 1}^2 / \sum [1 / (\delta z_{0i})^2]$$

where

$$(3.30-3) \quad e_{\sigma 1}^2 = [1 / (m - 1)] \sum [v_{0i}^2 / (\delta z_{0i})^2]$$

$$(3.30-4) \quad v_{0i} = z_{0i} - z_{00}$$

and δz_{0i} is the rms error of z_{0i} .

If, instead, δz_{0i} is chosen to represent the average deviation $e_{ai}(z_{0i})$ or the probable error $e_{pi}(z_{0i})$ of z_{0i} , then Eqs. (3.19-16) through (3.19-21) provide the means for comparable expressions of $e_{\sigma 0}(z_{00})$ or $e_{p0}(z_{00})$.

The rms error given by Eq. (3.30-2) is said to represent the *external consistency* of the measurements. It should be comparable in magnitude to the rms value

$$(3.30-5) \quad \left\{ \sum [1 / (\delta z_{0i})^2] \right\}^{-1/2}$$

if there were no constant or systematic errors in any of the determinations z_{0i} . The expression (3.30-5) is said to represent the *internal consistency* of the measurements.

If e_{σ_0} in Eq. (3.30-2) is many times larger than expression (3.30-5), there is reason to suspect that the experiments that yielded the z_{0i} differed from each other by the presence of one or more systematic errors in one or more of the experiments. (This criterion is known as Gauss's criterion.) Another phrasing is to say that if the differences among the means are much larger than the rms errors of the means, systematic errors may have gone undetected. Ordinarily, if $e_{\sigma_0}^2$ as given by Eq. (3.30-3) is greater than about 10, there is reason to suspect the presence of a systematic error in some of the determinations of z_0 .

3.31 Indirect measurements of several quantities. If m quantities $x_1, x_2, \dots, x_j, \dots, x_m$ cannot be measured directly, but if n functions $f_i(x_1, x_2, \dots, x_j, \dots, x_m)$ yield n measurements s_i ($i = 1, 2, \dots, n$), then the most probable values of the m quantities can be obtained by minimizing the sum of the appropriately weighted squares of the differences between each pair of f_i and s_i . If the weight of each measurement is w_i , it is required that

$$(3.31-1a) \quad S = \sum_{i=1}^n [f_i(x_1, x_2, \dots, x_j, \dots, x_m) - s_i]^2 w_i = \text{a minimum.}$$

Then the most probable values of the x 's are obtained by solution of the m simultaneous equations

$$(3.31-1b) \quad \phi_j \equiv \partial S / \partial x_j = 0 \quad (j = 1, 2, \dots, m).$$

This group of equations is termed the "set of normal equations."

The set of normal equations is generally difficult to solve unless the functions f_i are linear algebraic functions of the x 's. If they are not, it may be possible to find some function X of each x (like the logarithm, sine, or square) that does allow expression of the normal equations as a set of linear algebraic equations in the X 's.

If each f_i is a linear function of the x 's,

$$(3.31-2) \quad f_i = \sum_{j=1}^m (\alpha_{ij} x_j)$$

one may readily compute not only the most probable value of x_j but also the uncertainty $e_{\sigma}(x_j)$ in that value, assuming that the distribution of errors is Gaussian. One arranges the equations $\phi_j = 0$ in rows representing increasing values of j , and rearranges the equations so that they assume the form

$$(3.31-3) \quad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1j}x_j + \dots + a_{1m}x_m &= y_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2j}x_j + \dots + a_{2m}x_m &= y_2 \\ \dots & \\ a_{j1}x_1 + a_{j2}x_2 + \dots + a_{jj}x_j + \dots + a_{jm}x_m &= y_j \\ \dots & \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mj}x_j + \dots + a_{mm}x_m &= y_m \end{aligned}$$

The solution of this set of m simultaneous equations in m unknowns yields the m quantities x_j . Denote the determinant of the coefficients on the left side of Eqs. (3.31-3) as D , where

$$(3.31-4) \quad D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{vmatrix}$$

and denote the cofactor of any diagonal term a_{jj} as D_{jj} . Then the rms error e_o of x_j is given by

$$(3.31-5) \quad e_o^2(x_j) = (D_{jj}/D) \bullet e_{\sigma_j}^2$$

where

$$(3.31-6a) \quad (n - m)e_{\sigma_1}^2 = \sum_{i=1}^n w_i s_i^2 - \sum_{j=1}^m x_j y_j$$

or

$$(3.31-6b) \quad (n - m)e_{\sigma_1}^2 = \sum_{i=1}^n [f_i(x_1, x_2, \dots, x_m) - s_i]^2 w_i$$

Equations (3.31-6a) and (3.31-6b) are alternate expressions of the same quantity. The advantage of the first expression is that the two summations on the right side can be computed at the time that the original data are entered into any computer program; the disadvantage is that the right side represents a very small difference between two relatively large quantities so that each of these quantities must be computed with very high arithmetic accuracy. The advantage of the second expression is that it represents the direct summation of very small quantities, so that high computational accuracy is not required; the disadvantage is that the original data must be reentered into a computer program after the simultaneous equations have been solved.

Example 3.31. To calibrate a set of 1-, 2-, 3-, and 5-gram chemical-balance weights, there are available a 10-gram standard and a lower-quality weight of nominal 1-gram mass. Denote the masses of these weights, respectively, as M_1 , M_2 , M_3 , M_5 , M_{10} , and M_1' . Calibrations are effected by direct measurement of the differences between nominally equal combinations of weights. The following measurements are made:

$$(3.31-7a) \quad M_5 + M_3 + M_2 - M_{10} = S_1$$

$$(3.31-7b) \quad M_5 - (M_3 + M_2) = S_2$$

$$(3.31-7c) \quad M_3 - (M_2 + M_1) = S_3$$

$$(3.31-7d) \quad M_1 - M_1' = S_4$$

$$(3.31-7e) \quad M_3 - (M_2 + M_1') = S_5$$

$$(3.31-7f) \quad M_5 - (M_3 + M_1) + M_1' = S_6$$

Let

$$s_1 = S_1 + M_{10}'$$

Each measurement s_i is, *a priori*, assumed to have equal uncertainty δs and hence equal weight w_i . In this example, $n = 6$ and $m = 5$ (because M_1' is included), with x_1, x_2, x_3, x_4, x_5 , corresponding to M_5, M_3, M_2, M_1, M_1' , respectively.

Application of Eq. (3.31-1) leads to the set of normal equations

$$\begin{aligned}
 3M_5 - M_3 & & - M_1 - M_1' & = M_{10}' + S_1 + s_2 + s_6 \\
 -M_5 + 5M_3 & & & = M_{10}' + S_1 - s_2 + s_3 + s_5 - s_6 \\
 (3.31-8) & & 4M_2 + M_1 + M_1' & = M_{10}' + S_1 - s_2 - s_3 - s_5. \\
 -M_5 & & + M_2 + 3M_1 & = s_4 - s_3 - s_6 \\
 -M_5 & & + M_2 & + 3M_1' = -(s_4 + s_5 + s_6)
 \end{aligned}$$

which may be solved for M_1, M_2, M_3, M_5 , and, incidentally, for M_1' .

The value of D is 300. The diagonal terms of D are $a_{11} = 3, a_{22} = 5, a_{33} = 4, a_{44} = 3, a_{55} = 3$, and their respective cofactors are $D_{11} = 150, D_{22} = 66, D_{33} = 96, D_{44} = 134, D_{55} = 134$. Consequently, $e_{\sigma}^2(M_5) = (1/2)e_{\sigma 1}^2, e_{\sigma}^2(M_3) = 0.22e_{\sigma 1}^2, e_{\sigma}^2(M_2) = 0.32e_{\sigma 1}^2, e_{\sigma}^2(M_1) = e_{\sigma}^2(M_1') = (134/300)e_{\sigma 1}^2$, where $e_{\sigma 1}^2$ is given by Eq. (3.31-6).

In particular, if the values of S_1 and of s_2 through s_6 are, respectively, $-0.0025, -0.0015, -0.0075, -0.0035, -0.0115$, and -0.0045 , then $e_{\sigma 1}^2 = 8 \times 10^{-8}$ and the respective values of y_j, M_j , and $e_{\sigma}(M_j)$ are given by the following table:

y	9.9915	9.9845	10.0180	0.0085	0.0195
M	4.9980	2.9965	2.0030	1.0011 $\bar{6}$	(1.0048 $\bar{3}$)
$e_{\sigma}(M)$	0.0020	0.0001	0.0002	0.00019	(0.00019)

where the entries in the last column refer to the weight M_1' . (An overbar on a digit indicates that the digit repeats indefinitely.)

3.32 Application to an experimental measurement. The diagram of Fig. 3.32 shows how the results of this chapter may be applied to the measurement of a single physical variable. The numbers of the equations pertinent to the various steps are indicated. The words "probable error" may be replaced by "rms error" or "average error." The steps of the measurement operation are

1. Instruments having known probable errors are used to measure the experimental variables.
2. The measurement of each experimental variable is repeated n times to reduce the probable error by a factor of \sqrt{n} .
3. The mean value of each experimental variable, as computed, is inserted in a formula or other known relation that gives the value of the final physical variable in terms of the measured experimental variables.
4. The probable error of the final physical variable is deduced from the probable errors of the measured experimental variables.
5. If steps 1 through 4 have also been performed by other laboratories or on other occasions, the means deduced by each of the laboratories are weighted in accordance with the probable errors of each laboratory's results and then averaged to produce the Grand Mean.
6. The probable error of the Grand Mean is computed from the results which were used to derive it. This provides a measure of the external consistency of the data.
7. The probable error of the Grand Mean is computed on the assumption that only random instrument errors contributed to the Grand Mean. This provides a measure of the internal consistency of the data.
8. The errors obtained in steps 6 and 7 are compared. If the first is much larger than the second, there is reason to suspect that an unrecognized systematic error entered into the determinations by some of the laboratories.

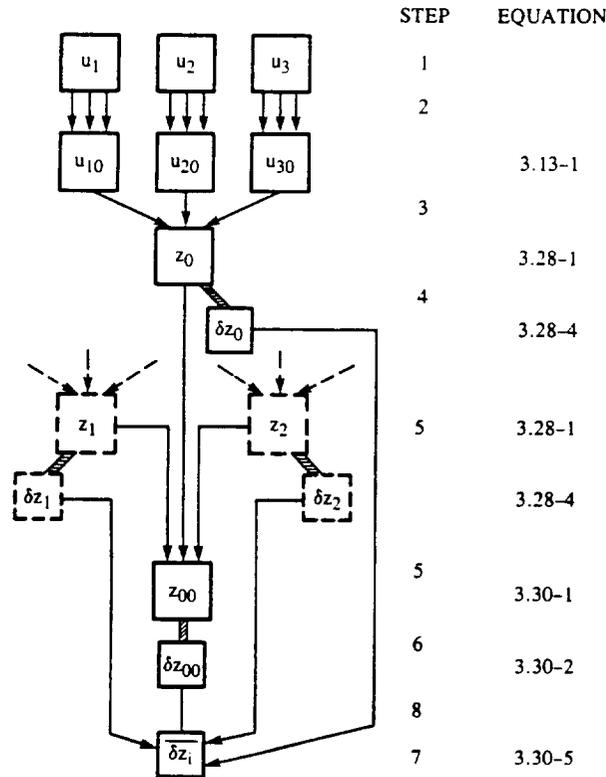


Figure 3.32.—Sequence of experimental measurements.

3.33 Calibration correction curves. A calibration correction curve is a graphical description of the systematic corrections to be applied to an instrument's indication. Such a curve for at least the first few units of a new instrument provides an understanding of the instrument's characteristics that may serve to determine the extent and nature of calibrations that should be made of subsequent units of the same type of instrument.

Most commonly, the curve takes the form of a plot of Δy versus y_{in} , where Δy is the correction to be added to the indication y_{in} in order to yield the true value y_{tr} of the quantity being measured. Only a small number of the pairs $(y_{in}, \Delta y)$ are usually determined; it is then necessary to decide how these points are to be joined in order to form a continuous curve. This decision usually cannot be made solely by inspection of the data; some knowledge of the nature of the instrument and of the calibration technique may also be required.

(1) If it is known that exactly the same value of Δy will be obtained each time that the instrument is subjected to the input y_{tr} , then the calibration curve may be drawn through each point $(y_{in}, \Delta y)$. Such a situation may exist if

- (a) the instrument and the determination possess negligible random error, or
- (b) Δy represents the mean of so large a number of observations that the rms error of the mean is negligible.

Example 3.33a. It is known that the manufacturer of a presumably linear 100-division instrument scale actually calibrates and marks the scale at eleven cardinal points and then machine divides each interval into ten subdivisions. Then if random errors are negligible, a calibration may be performed at the cardinal points, and then these points may be joined by straight-line segments.

Example 3.33b. It is known that the manufacturer of a digital millivoltmeter for a thermocouple-type pyrometer actually fits the thermocouple's temperature-emf curve with a number of straight-line segments. If the end points of these segments are known, calibrations at these same end points may be joined by straight-line segments.

(2) If it is known that the determination of Δy is subject to appreciable random error (due to the instrument or to the calibration technique), then a smooth curve should be faired through the points $(y_{in}, \Delta y)$.

Example 3.33c. If, in Example 3.33b, the end points of the segments are not known to the calibrator, and if the ability to match the temperature-emf curve is considered to be a possible source of significant error, then a smooth curve should be faired through the points.

Often, a straight-edge or French curve is sufficient to provide the desired calibration curve. If additional analysis of the curve is desired, the graphical and analytical techniques treated in the next chapter may be helpful; the analytical techniques are often needed if the data are to be entered into a computer program. The choice of a suitable technique, however, requires a human interpretation of the appearance of the calibration curve, based on an understanding of the physical operations involved.

3.34 The designation of procedures in the specification of accuracy. Many legal contracts for materials or performance presume a measurement accuracy that is higher than is practically achievable. The resulting difficulties may be minimized by agreement on all details of the apparatus and techniques that will be used in the measurement, and agreement to accept the resultant indication as "true." Such details would include the determination of the partial derivatives in Eq. (3.9-2) and the application of the resulting systematic corrections. The derivatives need be determined only once or only occasionally if similar sets of measurements will be made repeatedly over a long period of time.

Many "standards" of national and international organizations are concerned with establishing

- (1) standardized conditions and procedures for the measurement of a physical variable, and
- (2) standardized methods for correcting for deviations from the standardized conditions (e.g., deviations from standard pressure and temperature).

The resultant measurement represents "the value of the physical variable under the conditions of the specified standard."

The word "standard" may be replaced by the more precise term "recommended practice". On some occasions, this substitution has some legal advantage because it avoids the interpretation that the "standard" is mandatory.

3.35 Relation of accuracy to other factors affecting the selection of an instrument. Measurement accuracy is only one of many factors that affect the selection of a measurement technique and instrument. Some other factors are speed, reliability, convenience, and cost. Accuracy often drops when speed of measurement must be increased. Reliability of an instrument often increases as simplicity of instrument construction increases, even though such simplicity may involve some reduction in accuracy. Reduction of maintenance requirements, ease of repair and adjustment, and reduction of skills required for operation all affect reliability and overall cost; they may be related to the measurement accuracy that is obtainable. In some instances, the long-term availability of repair parts and services are also important considerations. The compromise among these factors depends on the weight assigned to each in any particular situation and depends on human judgment. Any formal mathematical procedures presented in this chapter are merely aids in the formulation of that judgment. Many of these procedures must be used with an appreciation of their approximate character, because they are based on mathematical assumptions chosen for their convenience rather than for their exactness.

NOTES FOR CHAPTER 3

N3.9. An even closer approximation is possible if the experiment is repeated using an additional change ($-\Delta x_i$); or using a larger change ($n \bullet \Delta x_i$).

(i) If the points on each side of (x_i, y_i) are $(x_i - \Delta x_i, y_i - \Delta y_1)$ and $(x_i + \Delta x_i, y_i + \Delta y_2)$, then

$$(\partial F / \partial x)_i = (\Delta y_1 + \Delta y_2) / (2 \bullet \Delta x_i)$$

(ii) If the points on each side of (x_i, y_i) are $(x_i - \Delta x_i, y_i - \Delta y_1)$ and $(x_i + n \bullet \Delta x_i, y_i + \Delta y_2)$, then

$$(\partial F / \partial x)_i = (n^2 \bullet \Delta y_1 + \Delta y_2) / [n(n + 1) \bullet \Delta x_i]$$

(This combination of points is useful at the extreme ends of a tabulation.)

N3.13 The molecular speeds depend only on the relative molar mass M and the gas temperature T . If R_0 is the universal gas constant ($8314.34 \text{ J} \bullet \text{kmol}^{-1} \bullet \text{K}^{-1}$) then the arithmetic mean speed v_a and the rms speed v_σ are given by

$$v_a^2 = 8R_0T / (\pi M); \quad v_\sigma^2 = 3R_0T / M.$$

N3.18 If the number n of observations is small, so that assumption G for the Gaussian distribution does not hold, the distribution assumes a form resembling that shown in Fig. 3.18(a) except that the peak amplitude at $e = 0$ is reduced, and the amplitudes at values of e/σ larger than about 1.6 are increased. This modified distribution is termed a *t-distribution*. Its distribution function is

$$(N3.18-1a) \quad f(t) = f_0 \bullet [1 + t^2 / (n - 1)]^{-n/2}$$

where

$$(N3.18-1b) \quad t = e / [(\sum v_i^2) / n]^{1/2}$$

$$(N3.18-1c) \quad f_0 = [(n - 1) \pi]^{-1/2} \Gamma(n/2) / \Gamma[(n - 1) / 2]$$

and v is given by Eq. (3.14-1).⁴

Equation (3.18-2) is replaced by a proportionality that depends on n :

$$(N3.18-2) \quad e_p : e_a : e_\sigma = (0.65 \text{ to } 0.67) : (0.93 \text{ to } 0.80) : 1$$

when n ranges from 4 to 32, respectively, and

$$(N3.18-3) \quad e_\sigma^2 = (2n - 1) / (2n - 3)$$

⁴In terms of factorials, the gamma function is given by

$$\Gamma(n/2) = [(n - 2)! \sqrt{\pi}] / [2^{n-2} (n-3)/2!] \quad \text{if } n \text{ is odd}$$

$$\Gamma(n/2) = [(n - 2) / 2]! \quad \text{if } n \text{ is even}$$

Table 3.18.2 lists values of t , for several values of n , at which the probability P of exceeding t is 2, 3, 5, 10, 15, or 20 percent. When $n > 4$ and $P \geq 5\%$, the listed value of e corresponding to a particular probability does not exceed the value listed for a Gaussian distribution, by more than the uncertainty in the measure of dispersion, as treated in Sec. 3.22. Hence, for $n > 4$, $P \geq 5\%$, use of the Gaussian distribution is adequate for all practical purposes.

N3.19. The distinction between the deviation of a single observation, which serves to describe past events, and the deviation of the mean, which serves to estimate future events, is illustrated by the following examples, in which the error distribution is neither Gaussian nor uniform.

(a) Suppose that n observations are made of a quantity y which is likely to have constant error δy and that half of the observations yield a value $y_0 + \epsilon$ and half of the observations yield a value $y_0 - \epsilon$, these two values occurring randomly. Then, if n is large, one may conclude that

- (i) The average value of y is y_0
- (ii) The average of the absolute value of the individual deviations is ϵ , and this average deviation is substantially independent of the number n of observations
- (iii) In the future, if there is no change in the circumstances of the experiment, one may expect that the true value of y will be y_0 with an uncertainty that decreases as n increases. (The uncertainty will be on the order of ϵ/\sqrt{n} .)

(b) If the quantity y had been likely to have constant fractional error $\delta y/y$ and if the observations had yielded values of $y_0(1 + \epsilon)$ and $y_0(1 - \epsilon)$ in equal number, then, in similar manner, one would conclude that

- (i) The average value of y is y_0
- (ii) The average fractional deviation of past observations is ϵ , substantially independently of n
- (iii) The fractional uncertainty in assuming y_0 to be the value of y in the future is about ϵ/\sqrt{n}

N3.27 A frequency meter is an instrument whose indication y is almost proportional to the time integral of its input Y . Its behavior is described by

$$y = (1/\tau) \int (Y - y) dt$$

where τ is the characteristic time constant of the instrument and the integration interval is presumed to be much longer than τ . This instrument is considered in Sec. 6.8.

N3.29 I. When there are two uniform distributions

$$-A \leq e_A \leq A \quad \text{and} \quad -B \leq e_B \leq B \quad (B \geq A > 0)$$

and

$$e = |e_A + e_B|; \quad f = dP/de$$

then

$$f = 1/(2B) \quad \text{for} \quad 0 \leq e \leq (B - A)$$

$$f = (A + B - e)/(4AB) \quad \text{for} \quad (B - A) \leq e \leq (B + A)$$

II. When there are three uniform distributions

$$-A \leq e_A \leq A; \quad -B \leq e_B \leq B; \quad -C \leq e_C \leq C; \quad (C \geq B \geq A > 0)$$

and

$$e = |e_A + e_B + e_C|; \quad f = dP/de$$

then

For $0 \leq e \leq (B + A - C)$,

$$f = 1/(2C) \quad \text{if} \quad C > (B + A)$$

$$f = [4AB - (A + B - C)^2 - e^2]/(8ABC) \quad \text{if} \quad C < (B + A)$$

For $(B + A - C) \leq e \leq (C - B + A)$,

$$f = [8AB - (A + B - C + e)^2]/(16ABC)$$

For $(C - B + A) \leq e \leq (C + B - A)$,

$$f = (B + C - e)/(4BC)$$

For $(C + B - A) \leq e \leq (C + B + A)$,

$$f = (C + B + A - e)^2/(16ABC)$$

III. In the special case when $A = B = C$ for all three uniform distributions,

$$\text{For } 0 \leq e \leq A, \quad f = (3A^2 - e^2)/(8A^3)$$

$$\text{For } A \leq e \leq 3A, \quad f = (3A - e)^2/(16A^3)$$

and $e_p = 0.71A$, $e_a = 0.81A$, $e_\sigma = A$.

CHAPTER 4. EMPIRICAL GRAPHS AND EQUATIONS

4.0 Introduction. Symbols. The material in this chapter is relevant principally to the interpretation and manipulation of experimental data, in order to improve the understanding of what is being studied, to plan the course of future operations and to monitor their progress, to analyze and to present results, and to present conclusions.

The following symbols are used frequently in this chapter.

e_a or a	average error or deviation
E_a	fractional average error or deviation
e_σ or σ	rms error (standard deviation)
E_σ	fractional rms error (fractional standard deviation)
m	number of independent constants
n	number of observations
r	correlation coefficient
S	sum of squares of residuals
V	fractional residual
v	residual
w	weight of observation or of square of residual
x	independent variable
y	dependent variable
ρ_x, ρ_y	index of correlation

Symbols used occasionally will be defined as they are introduced.

4.1 The uses of empirical graphs and equations. The complete understanding of an instrument's characteristics may require the determination of how the instrument is affected by several variables other than the principal variable that the instrument is to measure. Similarly, the complete understanding of a measurement technique may require the determination of the influence of each of several variables that influence the measurement. Quantitatively, these determinations may involve the determination of the form and magnitude of the partial derivatives that appeared in Eqs. (3.9-2) and (3.28-3).

Example 4.1a. The natural frequency of a tuning fork or crystal may be affected by its orientation and by the pressure, temperature, and composition of the medium in which it is immersed.

Example 4.1b. The indication of a thermometer bulb immersed in a flowing fluid is affected by the Reynolds number of the flow, the Prandtl number of the fluid, and the shape and physical properties of the bulb's support.

When a new experimental procedure is initiated, or a new type of instrument is tested, initial experiments seek to determine the systematic effects of one variable at a time. Dimensional analysis is a powerful tool in minimizing the number of independent variables that need to be identified. Initially, graphical methods

are effective in clarifying the nature of the relationships between dependent and independent variables. Once this identification has been made, analytical methods of describing the relationships may be used

- (i) to implement subsequent mechanized operations, like calibration, or
- (ii) to facilitate subsequent routine data analysis, or
- (iii) to improve understanding of the instrument and of the measurement technique and, thereby, to improve the operation to which they are applied.

Graphical methods help to monitor an experiment by identifying anomalies in the data or in the procedures (or, conversely, by confirming their reliability). Graphs, and equations that represent them, are also compact, easily comprehended means of presenting results. In this chapter, such empirical graphs and equations will be considered.

When the relationship between two variables x and y is determined by a few pairs of simultaneous measurements (x_i, y_i) over a range $x_1 \leq x \leq x_m$, and if $y(x)$ is continuous over this range, a graph or an analytical equation, that represents the most probable relationship between the two variables, offers two advantages:

- (i) It provides greater accuracy than does the mere tabulation of the values themselves, because all the data jointly contribute to the result.
- (ii) It permits interpolation between the pairs of values that represent the actual measurements.

The operation of fairing a smooth curve through a set of discrete points is, thus, analogous to taking the mean of a set of measurements of a single quantity or, more precisely, to determining the most probable value of a function of several variables.

4.2 Graphs. There is no mathematical law that will determine what type of curve should be drawn through a set of plotted points. The decision whether the curve should be a straight line, or a parabola, or whether it should have inflection points (and, if so, how many) must be made by the analyst on the basis not only of the appearance of the plotted data, but also of the analyst's understanding of the physical phenomena represented by the data. The decision becomes less difficult and more obvious as the number of data increases and as the scatter of data from an assumed curve decreases. Conversely, as the physics of the situation is better understood, fewer pairs of measurements are needed to establish the faired curve. The criteria used in fairing are also optional; usual human judgment is to minimize the sum of the absolute value of the distances of the plotted points from the faired curve. If rectangular Cartesian coordinates are used, these distances may be measured in a vertical direction, in a horizontal direction, or in a direction normal to the curve.

Once the form of the curve has been chosen, it may be convenient to represent it by an analytical formula. However, the criterion used to select the original graph may no longer be convenient for an analytical treatment; usually, considerations of mathematical convenience dictate that, to deduce an analytic formula, the least-squares criterion should be used and that it should be applied to only one of the variables. This subject will be treated more completely in later sections.

4.3 Equations. The operation of finding an analytical expression for the relationship between x and y is actually the operation of determining the most probable values of the constants of an assumed empirical equation. There is no mathematical procedure that will tell in advance what form of equation is followed by a given set of data. [Note N4.3.1] Mathematics can serve only to tell how closely an equation of assumed form fits the actual data. The selection of the form of the equation to be assumed requires human judgment; that judgment is made from a mental comparison of the faired curve with the graphs of known analytic functions, from a physical intuition concerning the relationship between x and y , or by trial and error. The judgment is also influenced by whether the range of x that is of interest does or does not extend beyond the range of the data.

Once the form of the equation has been selected, presumably for use with future sets of experimental data, the criterion for selecting the best fit (i.e., the most probable values of the constants in the equation) must be specified. The least-squares criterion is the most common choice because of its convenience—cne

minimizes the sum of the squares of the deviations of the measurements from the value predicted by the equation. Another criterion is the Chebyshev criterion—one minimizes the maximum deviation of the measurements from the value predicted by the equation. For any criterion, the range of the variables over which the criterion is to be applied must be specified. An equation which has been computed to provide the best fit over a specified range may not generally be used to extrapolate beyond that range [Note N4.3.2].

After the best-fit criterion has been chosen, some additional decisions must be made before the formal equation-fitting operation can be started. These decisions, discussed in Sec. 3.15, require assumptions of the relative weights to be assigned to the various experimental observations.

The trial selection of the form of equation to be fitted to a set of data is often made by inspection of the plot of $y(x)$ and by the pragmatic consideration of expediency of mathematical operations. For example,

1. if $y(x)$ is always positive and symmetrical about the ordinate $x = x_0$, then $y(x)$ should contain only even powers of $(x - x_0)$.

2. if $y(x)$ lies only in the first quadrant, is monotonic, and is concave upward, a formula of the form

$$(4.3-1) \quad y = \sum a_i x^i \quad (i = 0, 1, 2, \dots)$$

may be appropriate.

3. if $y(x)$ lies only in the first quadrant, is monotonic, and is concave downward, a formula of the form

$$(4.3-2) \quad y = \sum a_i (\sqrt{x})^i \quad (i = 0, 1, 2, \dots)$$

also may be appropriate.

4. if $y(x)$ lies in the first and third quadrants, is symmetrical about the origin, and is monotonic, an expression like Eq. (4.3-1), with only odd values of i , may be appropriate.

5. if $y(x)$ lies in the first quadrant, is monotonic, and appears to approach a horizontal asymptote, then

$$(4.3-3) \quad y = \left(\sum_0^m a_i x^i \right) / \left(1 + \sum_1^m b_i x^i \right)$$

is a general form that may be appropriate.

6. if $y(x)$ has a point of inflection, any assumed polynomial must be at least of the third degree.

7. if $y(0) = 0$, then a_0 should be zero in Eqs. (4.3-1) to (4.3-3).

4.4 The case of both variables subject to error. When both variables, x and y , are subject to error, the fitting of an appropriate graph poses no serious problem; a convenient criterion is to minimize the sum of the visually measured distances of the plotted points from the faired curve.

However, when the curve fitting is to be performed analytically, by means of some chosen form of equation, the required mathematical operations are usually complex and inconvenient, and require an advance estimate of the rms errors of each of the variables. It is more expedient to assume that only one of the variables is subject to error and that the measurements of the other variable are exact. This latter variable is chosen as the one which is believed to have the smaller fractional rms error. In this text, x will designate the variable which is assumed to be without error.

An important exception occurs when there is a linear relationship between x and y . It then remains convenient to treat both x and y as being subject to error and to describe mathematically the *correlation* between the two variables. Correlation is treated in Sec. 4.12.

4.5 Criteria for best fit of a curve or equation. The fitting of a curve or equation to a set of measured points (x_i, y_i) depends on what prior assumption is made concerning the relative reliability, or *weight*, of the individual points and, as a practical matter, on whether graphical or analytical methods are chosen.

In fairing a *curve* through plotted points, one seeks to minimize the sum of the visually measured distances of the points from the curve, treating all distances as positive. The operation is analogous to taking the

arithmetic mean of a set of data. The nature of the coordinate system and the relative fractional accuracy of x and y determine whether the distances will be measured in the direction of x , or of y , or of the normal to the curve at that point. For example, if rectangular Cartesian coordinates are used, then

- (a) If the *a priori* uncertainties in x and y are represented by distances of comparable magnitude, deviations should be measured along the normals to the curve.
- (b) If the *a priori* uncertainty in y is represented by a much larger distance than is the uncertainty in x , deviations should be measured in the direction of y .

A graphical construction is particularly advantageous when both x and y may have appreciable errors.

In fitting an *equation* to a set of points, mathematical convenience requires that x be presumed to be without error. One then seeks to minimize the sum of the appropriately weighted squares of the differences between the measured value and the value predicted by the equation. These differences will be termed *residuals*, as in Sec. 3.14, so that if the fitted equation is $y = f(x)$, then, for any point (x_i, y_i) ,

$$(4.5-1) \quad v_i = f(x_i) - y_i.$$

Case 1. If each y_i is presumed to have substantially the same absolute uncertainty ($\delta y = \text{constant}$), then we seek f such that

$$(4.5-2) \quad \sum v_i^2 = \text{a minimum.}$$

(All summations not otherwise identified are over the index i .)

Case 2. If each observation y_i is presumed to have the weight w_i , then we seek f such that

$$(4.5-3) \quad \sum v_i^2 w_i = \text{a minimum.}$$

Case 3. If each observation y_i is presumed to have the same fractional uncertainty ($\delta y/y = \text{constant}$), then the weight of each observation is $1/y_i^2$, so that we seek f such that

$$(4.5-4) \quad \sum v_i^2 (1/y_i^2) = \text{a minimum.}$$

In an alternate phrasing, we could have defined a relative residual

$$(4.5-5) \quad V_i = [f(x_i) - y_i]/y_i$$

and sought f such that

$$(4.5-6) \quad \sum V_i^2 = \text{a minimum.}$$

Equations (4.5-1) and (4.5-5) are comparable to Eqs. (3.14-1) and (3.19-8).

If there are n pairs of data ($i = 1, 2, \dots, n$) and if $f(x)$ has m constants whose most probable values are to be determined, then a mathematical measure of how well $f(x)$ agrees with the measured points is given by $e_{\sigma 1}$ or $E_{\sigma 1}$, where

$$(4.5-7a) \quad (n-m)e_{\sigma 1}^2 = \sum v_i^2 \quad \text{for Case 1}$$

$$(4.5-7b) \quad (n-m)e_{\sigma 1}^2 = \sum v_i^2 w_i \quad \text{for Case 2}$$

$$(4.5-7c) \quad (n-m)E_{\sigma 1}^2 = \sum V_i^2 \quad \text{for Case 3}$$

$$= \sum (v_i^2/y_i^2) \quad \text{for Case 3}$$

Note that the dimensions of $e_{\sigma 1}$ are different in the three cases. In Case 1, Eq. (4.5-7a), the dimensions of $e_{\sigma 1}$ are the dimensions of y ; in Case 2, Eq. (4.5-7b), the dimensions of $e_{\sigma 1}$ are the dimensions of $y\sqrt{w}$; in Case 3, Eq. (4.5-7c), $E_{\sigma 1}$ is dimensionless.

In the case of graphs, the definition of Eq. (4.5-1) is usable when uniformly spaced rectangular coordinates are used; then, if all y_i are presumed to have the same absolute uncertainty, the nature of human perception requires that the criterion (4.5-2) for Case 1 be replaced by

$$(4.5-8) \quad \sum |v_i| = \text{a minimum.}$$

When logarithmic coordinates are used, and all y_i are presumed to have the same fractional uncertainty, the definition of Eq. (4.5-5) is usable and the criterion (4.5-6) for Case 3 must be replaced by

$$(4.5-9) \quad \sum |V_i| = \text{a minimum.}$$

Criteria (4.5-8) and (4.5-9) for graphs usually do not produce results that are significantly different than those that would be produced by the criteria (4.5-2 and 4.5-6) that they replace.

Case 2 cannot usually be handled satisfactorily by graphical means if w_i is neither constant (so that uniformly spaced rectangular coordinates may be used) nor proportional to $1/y^2$ (so that logarithmic coordinates may be used), unless special graph paper is employed. An exception is provided by the use of error bars or rectangles, as described in Sec. 4.7.

Approximate formulas, comparable to Eq. (4.5-7), that provide a measure of how well the graph and the original data agree with each other, are

$$(4.5-10a) \quad n \bullet e_{a1} = \sum |v_i|$$

$$(4.5-10b) \quad n \bullet E_{a1} = \sum |V_i|$$

where e_{a1} and E_{a1} are termed *average errors*.

There remains the question of how to choose among various forms of the function f , if an equation is to be fitted, or among various shapes of curves, if a graphical solution is sought. There is no rigorous mathematical answer to this question; the answer is subjective and depends on the analyst's understanding of the physics of the phenomenon being studied and of the quality of the experiment. It is commonly felt that the simplest form of equation or curve should be sought. Gauss's criterion was to choose that form of f which yielded a minimum value of $e_{\sigma 1}$ or $E_{\sigma 1}$ for a given m . Ordinarily, $e_{\sigma 1}$ can be made smaller by making m larger. Obviously, in the extreme condition, $e_{\sigma 1}$ can be made zero if m is chosen equal to n ; the curve then passes through each datum, with the implication that each measurement is perfectly accurate. Some practical rules of thumb that have been used are that a larger value of m should not be chosen unless $e_{\sigma 1}$ or $E_{\sigma 1}$ is thereby reduced by a factor of at least 1.5, and that m should not exceed $n/4$ unless previous experience has demonstrated that f can be reliably defined by fewer than $4m$ measurements (e.g., if f is known to be a straight line).

4.6. Accuracy of a curve or equation. The quantities $e_{\sigma 1}$ or $E_{\sigma 1}$ in Eq. (4.5-7) are measures of how well the curve or equation agrees with the measured data already obtained. They have clear, simple meanings only when each measurement is deemed to have equal uncertainty (Case 1) or equal fractional uncertainty (Case 3). Their practical meaning may not be obvious if the measurements have other weights.

However, in future use of the curve or equation to estimate the most probable value of $y(x)$ for any given x , the uncertainty in $y(x)$ is given by $e_{\sigma 0}(y)$ or $E_{\sigma 0}(y)$, where

$$(4.6-1a) \quad e_{\sigma 0}^2(y) = e_{\sigma 1}^2/n \quad \text{for Case 1}$$

$$(4.6-1b) \quad e_{\sigma 0}^2(y) = e_{\sigma 1}^2/\sum w_i \quad \text{for Case 2}$$

$$(4.6-1c) \quad E_{\sigma 0}^2(y) = E_{\sigma 1}^2/n \quad \text{for Case 3}$$

The dimensions of $e_{\sigma 0}(y)$ are always the dimensions of y ; $E_{\sigma 0}(y)$ remains dimensionless. The quantity $e_{\sigma 0}(y)$ has a clear, simple meaning even for weighted observations of any type.

Whereas $e_{\sigma 1}$ and $E_{\sigma 1}$ hardly change when n increases considerably, the measures $e_{\sigma 0}(y)$ or $E_{\sigma 0}(y)$ decrease in proportion to \sqrt{n} in Cases 1 and 3, or to $\sqrt{(\sum w_i)}$ in Case 2. Thus, the accuracy of future estimated values of $y(x)$ increases appreciably with the number of data that were used to determine $f(x)$ originally.

By similar reasoning, the uncertainty in future estimated values of y by means of a graph is given by $e_{\sigma 0}(y)$ or $E_{\sigma 0}(y)$, where

$$(4.6-2a) \quad e_{\sigma 0}(y) = e_{\sigma 1}/\sqrt{n}$$

$$(4.6-2b) \quad E_{\sigma 0}(y) = E_{\sigma 1}/\sqrt{n}$$

and where $e_{\sigma 1}$ and $E_{\sigma 1}$ are given by Eq. (4.5-10).

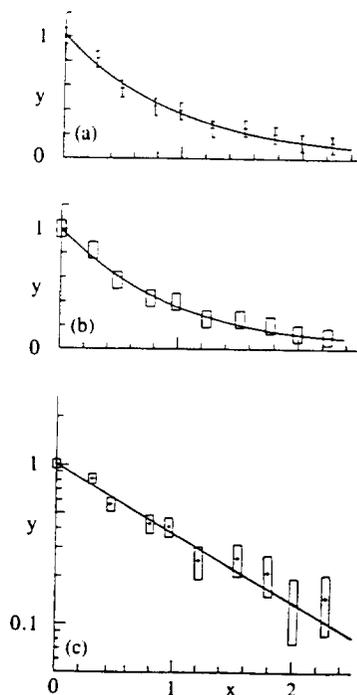
4.7 Error bars and rectangles. The graphical construction of a curve through a set of points plotted on rectangular coordinates may be facilitated by constructing, around each point, a rectangle whose width and height represent the estimated probable, average, or rms uncertainties in x and y at that point. If only y has appreciable error, the rectangle is replaced by a vertical bar. The simplest curve that passes through all the rectangles or bars is likely to provide an adequate representation of the relation between x and y . The generalization to other coordinate systems, such as the polar, is apparent. The size of the error rectangle or bar varies inversely with the weight which that point should have in defining the most fitting curve.

An error bar may be used for a different purpose in the graphical presentation of results obtained by graphical or analytical means. It may be erected at each measurement point, or merely at one point anywhere along the graph, to represent the value of $e_{\sigma 1}(y)$ or $E_{\sigma 1}(y)$ deduced from the operations treated in Sec. 4.5. In this usage, the bar represents a conclusion rather than an *a priori* assumption.

Example 4.7. An oscillograph is used to record the decay of voltage E , from an initial value E_0 , when a capacitor C is discharged through a resistor R . The applicable physical law is $E = E_0 \exp[-t/(RC)]$. The data are to be replotted by using the dimensionless variables $y = E/E_0$ and $x = t/(R_1 C_1)$, where R_1 and C_1 are the nominal values of R and C . The *a priori* uncertainty in E is $0.06 E_0$.

If the *a priori* uncertainty in x is negligible, and the data are plotted on uniformly spaced rectangular coordinates, the graph will appear as in Fig. 4.7(a), wherein each point is represented by a vertical error bar. If there is also an uncertainty in x of magnitude 0.04., the graph will appear as in Fig. 4.7(b), wherein each point is represented by an error rectangle. The advantage of the constructions of Fig. 4.7(a) and Fig. 4.7(b) is that each point has equal weight because each error bar or error rectangle is of the same size.

If the uncertainty in x is 0.04, and the data are plotted on semilog paper, the graph will appear as in Fig. 4.7(c). The advantage of this construction is that it is known in advance that the graph should be a straight line. However, at small values of y , the vertical height of error bars or rectangles becomes large and also asymmetrical about the plotted point.



(a) $\delta x = 0, \delta y = \text{constant}$
 (b) $\delta x = \text{constant}, \delta y = \text{constant}$
 (c) $\delta x = \text{constant}, \delta y = \text{constant}$

Figure 4.7.—Uses of error bars and rectangles.

4.8 Linearized graphical constructions. It is usually advantageous to deal with a linear relation between the independent and dependent variables, because then moving a straightedge about on a sheet of graph paper permits visual estimation of the most fitting straight line. If a linear relation does not exist, then transformation of variables may serve to produce a linear relationship. Such transformations must always be used with full consideration of the changes they make in the error rectangles or bars.

To facilitate the use of linear relations between transformed variables, special graph papers are available. In addition to the common-logarithm scales, there are also papers which provide a scale of reciprocals, of the logarithm of the logarithm, or of Gaussian probability ($\int f(e) \cdot de$, where $f(e)$ is given by Eq. (3.18-1)).¹

A particular advantage of the use of such a special scale is that it provides a guide to the planning of experiments: it is desirable that data be taken at values of the independent variable that are equally spaced on the scale representing that variable.²

Some examples of the transformation of variables to yield a linear relationship are listed here:

1. If the variables can be arranged or transformed to produce a straight line on uniformly spaced rectangular coordinate paper, the y -axis intercept b and the slope m define the relation

$$(4.8-1) \quad y = mx + b.$$

¹For example, the kinematic viscosity ν of lubricating oils as a function of temperature T often forms a straight line on graph paper whose coordinates are proportional to $\log T$ and to $\log \log \nu$.

²Some experimenters prefer to have slightly closer spacing near the ends of the range of measurement.

2. If the variables can be arranged or transformed to produce a straight line on semilog paper [Note N4.8] with uniformly spaced x -coordinates, and if

- (a) the intercept on the log y axis is log B ,
- (b) x_1 and x_2 are the abscissa of two points for which $y(x_2) = 10 \cdot y(x_1)$,

then

$$(4.8-2a) \quad y = B \exp(x/M)$$

where

$$(4.8-2b) \quad M = (x_2 - x_1) / \ln 10.$$

For example, in Fig. 4.7(c), $B = 1$ and $M = -1$.

3. If the variables can be arranged or transformed to produce a straight line on log paper whose horizontal and vertical decades are of equal length, and if

- (a) the line passes through the point $(\log x_1, \log y_1)$
(the scale markings for this point are x_1 and y_1)
- (b) the slope M of this line is taken as the ratio between the lengths (in cm or other unit of length) of the vertical and horizontal legs of a right triangle erected on the line with the line as hypotenuse,

then

$$(4.8-3) \quad y = (y_1/x_1^M) \cdot x^M.$$

4. If the intercept y_0 on the y -axis of rectangular Cartesian coordinates is clearly defined, and there is also a clearly defined horizontal asymptote y_m , one of the transformations

$$(4.8-4a) \quad Y = (y - y_0) / (y_m - y_0)$$

or

$$(4.8-4b) \quad Y = (y - y_m) / (y_0 - y_m)$$

may lead to a linear relationship between x and Y .

5. If the intercept x_0 on the x -axis is clearly defined, and there is also a clearly defined vertical asymptote x_m , one of the transformations

$$(4.8-5a) \quad X = (x - x_0) / (x_m - x_0)$$

or

$$(4.8-5b) \quad X = (x - x_m) / (x_0 - x_m)$$

may lead to a linear relationship between X and y .

6. If $y(x)$ on uniformly spaced rectangular coordinates is monotonic, has no inflections, and approaches a horizontal asymptote as x decreases, plot $Y = 1/y$ versus x on uniformly spaced rectangular coordinates. If a straight line of the form

$$Y = mx + b$$

results, then

$$(4.8-6a) \quad y = ax/(1 + cx)$$

where

$$(4.8-6b) \quad a = 1/b; \quad c = ma.$$

7. If $y(x)$ on uniformly spaced rectangular coordinates passes through the origin, is monotonic, has no inflections, and approaches a horizontal asymptote as x increases, plot $Y = 1/y$ versus $X = 1/x$ on uniformly spaced rectangular coordinates. If a straight line of the form

$$Y = mX + b$$

results, then

$$(4.8-7) \quad y = x/(bx + m).$$

8. If a plot on semilog paper (with logarithmic y -scale) shows only a slight curvature, and there are two points (x_1, y_1) and (x_3, y_3) near the extremities of the curve which have good accuracy,

- (a) compute $x_2 = (x_1 + x_3)/2$,
- (b) find $y_2 = y(x_2)$ on the curve,
- (c) compute $y_0 = (y_1 y_3 - y_2^2)/(y_1 + y_3 - 2y_2)$.

Then a plot of $y - y_0$ versus x on semilog paper may yield a straight line. If it does, follow example 2 of this listing.

9. If a plot on log paper shows only a slight curvature and there are two points (x_1, y_1) and (x_3, y_3) near the extremities of the curve which have good accuracy,

- (a) compute $y_2 = \sqrt{(y_1 y_3)}$,
- (b) find $x_2 = x(y_2)$ on the curve,
- (c) compute $x_0 = (x_1 x_3 - x_2^2)/(x_1 + x_3 - 2x_2)$.

Then a plot of y versus $x - x_0$ on log paper may yield a straight line; if it does, follow example 3 of this listing.

Alternatively,

- (a) compute $x_2 = \sqrt{(x_1 x_3)}$,
- (b) find $y_2 = y(x_2)$ on the curve,
- (c) compute $y_0 = (y_1 y_3 - y_2^2)/(y_1 + y_3 - 2y_2)$,

and, if a plot of $y - y_0$ versus x on log paper yields a straight line, follow example 3 of this listing.

10. If $y(x)$ plotted on uniformly divided rectangular coordinates passes through the origin and shows only moderate, monotonic deviation from the tangent at the origin, a plot of $Y = y/x$ versus x may yield a straight line

$$Y = Mx + B.$$

Then

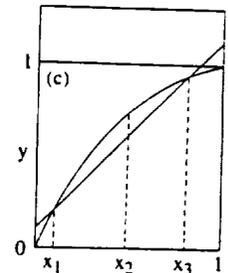
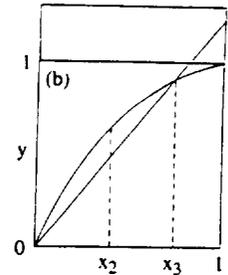
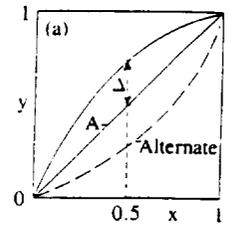
$$(4.8-8) \quad y = Bx + Mx^2.$$

4.9 Linear approximations of a quadratic. Measures of nonlinearity. Many nominally linear instruments (i.e., instruments whose indication is supposed to vary linearly with the quantity being measured) actually show a slight nonlinearity, so that the relation between the variables is more closely represented by a quadratic

TABLE 4.9.—PARAMETERS OF SIX METHODS OF DESCRIBING NONLINEARITY

Parameter	Method					
	Chord	Zero-based		Minimum error		
	A	B (min. LE)	C (min. e_o)	D (min. e_o)	E (min. LE)	F (min. e_a)
m	1	$1 + (12 - 8\sqrt{2})\Delta$	$1 + \Delta$	1	1	1
b	0	0	0	$2\Delta/3$	$\Delta/2$	$3\Delta/4$
x_1				$(\sqrt{3}-1)/6$	$(\sqrt{2}-1)/4$	$1/4$
x_2	$1/2$	$\sqrt{2}-1$	$3/8$	$1/2$	$1/2$	$1/2$
x_3		$2(\sqrt{2}-1)$	$3/4$	$(\sqrt{3}+1)/6$	$(\sqrt{2}+1)/4$	$3/4$
$ v(0) $	0	0	0	$2\Delta/3^{**}$	$\Delta/2^{**}$	$3\Delta/4^{**}$
$ v(x_2) $	Δ^{**}	$(12-8\sqrt{2})\Delta \approx 0.69\Delta^{**}$	$9\Delta/16^{**}$	$\Delta/3$	$\Delta/2^{**}$	$\Delta/4$
$ v(1) $	0	$(12-8\sqrt{2})\Delta \approx 0.69\Delta^{**}$	Δ	$2\Delta/3^{**}$	$\Delta/2^{**}$	$3\Delta/4^{**}$
e_o^2	$8\Delta^2/15$	$16(78-55\sqrt{2})\Delta^2/15$	$\Delta^2/5$	$8\Delta^2/90$	$7\Delta^2/60$	$23\Delta^2/240$
e_o	$\approx 0.73\Delta$	$\approx 0.48\Delta$	$\approx 0.45\Delta$	$\approx 0.30\Delta$	$\approx 0.34\Delta$	$\approx 0.31\Delta$
e_a	$2\Delta/3$	$4(37\sqrt{2}-52)\Delta/3$	$19\Delta/48$	$4\sqrt{3}\Delta/27$	$(2\sqrt{2}-1)\Delta/6$	$\Delta/4$
e_a	$\approx 0.67\Delta$	$\approx 0.43\Delta$	$\approx 0.40\Delta$	$\approx 0.26\Delta$	$\approx 0.30\Delta$	0.25Δ

** Limit of error (LE)



(a) Chord method
(b) Zero-based methods
(c) Minimum-error methods

Figure 4.9.—Methods of describing nonlinearity.

equation. In such cases, the manufacturer or the user may choose, for convenience, to approximate the relation by a straight line, and to provide some measure of the limit of error (or average error, or rms error) that may be expected by use of the approximation. The description of the error may be a statement like "nonlinearity does not exceed p percent of span." Some of the criteria that are commonly used to select the straight line approximation will be treated here, as well as the measures of nonlinearity that are associated with them.

It will be assumed that the relation between the variables is represented by a quadratic curve that passes through the points (X_0, Y_0) and (X_m, Y_m) at the extremes of the instrument's range. To a user, it is desirable that X represent instrument indication and Y represent the physical variable that the indication is supposed to represent. However, a manufacturer may prefer to reverse the roles of X and Y . In any event, X will be treated here as the independent variable. For convenience of presentation, we introduce the transformations

$$(4.9-1a) \quad x = (X - X_0)/(X_m - X_0) \quad 0 \leq x \leq 1$$

$$(4.9-1b) \quad y = (Y - Y_0)/(Y_m - Y_0) \quad 0 \leq y \leq 1$$

so that the relation between x and y appears as shown in Fig. 4.9 and

$$(4.9-2) \quad y(x) = x + 4x(1 - x)\Delta$$

where Δ is the difference, at abscissa $x = 1/2$, between $y(x)$ and the chord through the end points; Δ is also the maximum difference between the curve and the chord.

The quadratic parabola of Eq. (4.9-2) may be approximated by a straight line whose general equation is

$$(4.9-3) \quad y = mx + b$$

where the parameters m and b are chosen in accordance with some criteria selected by the instrument user. Each straight line leads to a residual

$$(4.9-4) \quad v(x) = y(x) - (mx + b)$$

as the difference between $y(x)$ in Eq. (4.9-2) and the chosen straight-line approximation, Eq. (4.9-3). Table 4.9, together with Fig. 4.9, shows six different approximations that may be used, the salient features of each, and the values of the various measures of nonlinearity that describe how well the approximation resembles the actual quadratic parabola. These measures of nonlinearity usually take one of the following forms:

- (a) A statement of the maximum value of $|v|$, the limit of error (LE), in the range of the instrument
- (b) A statement of the average value e_a of $|v|$ over the range of the instrument
- (c) A statement of the rms value e_o of v over the range of the instrument.

As indicated in Sec. 3.5, each of these measures can be expressed in units of y , or as a percentage of the instruments' span or full-scale value. When the dimensionless representation of Eq. (4.9-1) is used, all three measures are necessarily dimensionless.

The six approximations treated in Table 4.9 fall into one of three categories:

- (I) Approximation A, represented in Fig. 4.9(a), is a chord specified to pass through both end points. A specification that reads "the deviation of the actual calibration curve from a straight line through the end points shall not exceed 1 percent of the instrument's span" is more difficult to meet than if the "1 percent" had been used in any of the other five approximations. Consequently, this specification leads to a higher instrument cost.
- (II) Approximation D, E, or F, represented in Fig. 4.9(c) by a straight line parallel to the chord through the end points, but offset vertically by an amount that depends on the criterion chosen, places the fewest demands on the manufacturer, thereby minimizing costs, but requires that the user incorporate both parameters m and b into his mathematical analysis of the data. The actual curve and the approximation intersect at abscissae x_1 and x_3 , where v is zero. Maximum positive and negative values of the residual occur at $x = x_2 = 1/2$, at $x = 0$, and at $x = 1$.
- (III) Approximations B and C, represented in Fig. 4.9(b), are each a straight line through the origin. They lie between categories I and II in the demands placed upon the manufacturer and the user. The actual curve and the approximation intersect at abscissa $x = x_3$, where v is zero. Maximum positive and negative values of the residual occur at $x = 1$ and at some intermediate abscissa $x = x_2$.

4.10 Analytic determination of the constants of an equation. The most probable values of the constants $a_1, a_2, \dots, a_j, \dots, a_m$ of an assumed equation relating several variables x, y, z, \dots can be determined by purely analytic means, if the criterion for defining the most probable values is defined. This criterion is usually taken as the requirement that the sum of the appropriately weighted squares of the residuals shall be a minimum; the procedure is then termed "the method of least squares." Some other criteria were illustrated in Sec. 4.9, in application to an assumed linear relationship. When a nonlinear formula is assumed, application of these other criteria often becomes awkward, while use of the method of least squares remains mathematically convenient. This convenience is a reason for the method's popularity. It will be used exclusively in the applications that will henceforth be treated.

If a measured variable s is dependent on a number of independent variables x, y, z, \dots and a number of as yet undetermined constants a_j (where $j = 1, 2, \dots, m$) so that

$$(4.10-1) \quad s = F(x, y, z, \dots, a_1, a_2, \dots, a_j, \dots, a_m)$$

and if the weight of each measurement s_i of s is w_i , then the least-squares method is used to determine the values of the a_j which make

$$(4.10-2) \quad S = \sum (F_i - s_i)^2 w_i = \text{a minimum.}$$

Here

$$(4.10-3) \quad F_i \equiv F(x_i, y_i, z_i, \dots, a_1, a_2, \dots, a_j, \dots, a_m)$$

and the summation is for $i = 1, 2, \dots, n$. It is necessary that

$$n \geq m.$$

(The equality sign represents the condition $S = 0$ because the number of undetermined constants equals the number of measurements and, hence, of equations.)

The general method of solution, which parallels that of Sec. 3.31, is to create and to solve the set of m simultaneous equations

$$(4.10-4a) \quad \varphi_j \equiv \partial S / \partial a_j = 0 \quad (j = 1, 2, \dots, m)$$

which is equivalent to the set

$$(4.10-4b) \quad \sum F_i \bullet (\partial F / \partial a_j)_i w_i = \sum s_i \bullet (\partial F / \partial a_j)_i w_i$$

This set is termed the "set of normal equations".

In the following sections, this general method will be applied to some simple situations that permit explicit computation of the a_j and of the associated measures of uncertainty. In particular, for each assumed form of the function F , three cases will usually be considered:

Case A. The *a priori* uncertainty δs in s is a constant. Then $w_i = 1$ for all i , and it is necessary that

$$(4.10-5) \quad S_A = \sum (F_i - s_i)^2 = \text{a minimum.}$$

Case B. The *a priori* fractional uncertainty $\delta s/s$ in s is a constant. Then the weight of each term in the summation (4.10-2) is $1/s_i^2$ and it is necessary that

$$(4.10-6) \quad S_B = \sum [(F_i/s_i)^2 - 1] = \text{a minimum.}$$

Case C. The weight w_i of each measurement s_i is other than unity or $1/s_i^2$. Then it is necessary that

$$(4.10-7) \quad S_C = \sum (F_i - s_i)^2 w_i = \text{a minimum.}$$

For each form of F , formulas for the following quantities will be given:

1. The most probable value of each independent a_j
2. The rms value $e_{\sigma 0}$ or $E_{\sigma 0}$ of the uncertainty (absolute, or fractional) that may be expected, in future, by use of the deduced a_j , assuming that the circumstances affecting future measurements are not different from those that governed the original measurements. The values of $e_{\sigma 0}$ and $E_{\sigma 0}$ will usually be expressed in terms of an intermediate parameter $e_{\sigma 1}$ or $E_{\sigma 1}$, which happens to be a measure of the difference between the original data and the value of s calculated by using the deduced a_j 's; it also happens to be Gauss's criterion for the appropriateness of the chosen functional form of F (a smaller value implies greater suitability). [Notes N4.10.1 and N4.10.2]

It is important to understand that the method of computation of these formulas is chosen for its mathematical convenience and is based on the following assumptions:

1. Values of x are without error.
2. The residuals, v (Eq. 4.5-1) or V (Eq. 4.5-5), are so small that they may be treated like differentials (dy or $d(\ln y)$), and the methods of the differential calculus may be used.
3. The distribution of the errors (δy or $\delta(\ln y)$) is Gaussian.

The mathematical exactness of the formulas, therefore, does not necessarily imply a comparable exactness in the description of a practical engineering situation; ordinarily, the formulas merely provide an adequate and useful approximate description.

4.11 Examples of least-squares solutions for polynomials. The succession of these examples will represent situations of increasing complexity. Each solution is derived by following the general procedure indicated in Sec. 4.10. If

- (a) pairs of measurements (x_i, s_i) are taken;
- (b) the relationship $s = F(x)$ is assumed; and
- (c) measurements of x are assumed to have negligible error;

solutions will be given, usually for all three cases A, B, and C, as described in Sec. 4.10.

Unless otherwise noted, the following symbolism will be used (all summations are for $i = 1, 2, \dots, n$):

For Case A (the *a priori* uncertainty δs in s is constant)

$$(4.11-1) \quad b_k = \sum x_i^k; \quad c_k = \sum x_i^k s_i; \quad d_k = \sum x_i^k s_i^2; \quad S_A = \sum [F(x_i) - s_i]^2$$

For Case B (the *a priori* fractional uncertainty $\delta s/s$ in s is constant),

$$(4.11-2) \quad B_k = \sum (x_i^k/s_i^2); \quad C_k = \sum (x_i^k/s_i); \quad D_k = \sum x_i^k; \quad S_B = \sum \{[F(x_i)/s_i] - 1\}^2$$

For Case C (each measurement s_i has weight w_i),

$$(4.11-3) \quad b_k = \sum x_i^k w_i; \quad c_k = \sum x_i^k s_i w_i; \quad d_k = \sum x_i^k s_i^2 w_i; \quad S_C = \sum [F(x_i) - s_i]^2 w_i$$

Example 4.11a. If

$$F(x) = a_1 x$$

then, in Case A, with symbolism in (4.11-1),

$$a_1 = c_1/b_2; \quad e_{\sigma}^2(a_1) = e_{\sigma_1}^2/b_2; \quad n e_{\sigma_0}^2(s) = e_{\sigma_1}^2$$

where

$$(n-1)e_{\sigma_1}^2 = d_0 - a_1 c_1$$

$$= S_A$$

[Note N4.11]

In Case B, with symbolism in (4.11-2),

$$a_1 = C_1/B_2; \quad e_{\sigma}^2(a_1) = E_{\sigma_1}^2/B_2; \quad n E_{\sigma_0}^2 = E_{\sigma_1}^2$$

where

$$(n-1)E_{\sigma_1}^2 = D_0 - a_1 C_1 = n - a_1 C_1$$

$$= S_B$$

[Note N4.11]

In Case C, with symbolism in (4.11-3),

$$a_1 = c_1/b_2; \quad e_\sigma^2(a_1) = e_{\sigma_1}^2/b_2; \quad b_0 \bullet e_{\sigma_0}^2(s) = e_{\sigma_1}^2$$

where

$$(n-1)e_{\sigma_1}^2 = d_0 - a_1c_1 \\ = S_C$$

[Note N4.11]

Example 4.11b. If

$$F(x) = a_0 + a_1x$$

then, in Case A, with symbolism in (4.11-1),

$$a_0 = (b_2c_0 - b_1c_1)/D = (c_0 - a_1b_1)/n; \quad e_\sigma^2(a_0) = e_{\sigma_1}^2 \bullet b_2/D$$

$$a_1 = (nc_1 - b_1c_0)/D = (c_0 - a_0n)/b_1; \quad e_\sigma^2(a_1) = e_{\sigma_1}^2 \bullet n/D$$

$$D = nb_2 - b_1^2; \quad ne_{\sigma_0}^2(s) = e_{\sigma_1}^2$$

where

$$(n-2)e_{\sigma_1}^2 = d_0 - (a_0c_0 + a_1c_1) \\ = S_A$$

[Note N4.11]

In Case B, with symbolism in (4.11-2),

$$a_0 = (B_2C_0 - B_1C_1)/D = (C_0 - a_1B_1)/n; \quad e_\sigma^2(a_0) = E_{\sigma_1}^2 \bullet B_2/D$$

$$a_1 = (nC_1 - B_1C_0)/D = (C_0 - a_0n)/B_1; \quad e_\sigma^2(a_1) = E_{\sigma_1}^2 \bullet n/D$$

$$D = B_0B_2 - B_1^2; \quad nE_{\sigma_0}^2(s) = E_{\sigma_1}^2$$

where

$$(n-2)E_{\sigma_1}^2 = n - (a_0C_0 + a_1C_1) \\ = S_B$$

[Note N4.11]

In Case C, with symbolism in (4.11-3),

$$a_0 = (b_2c_0 - b_1c_1)/D = (c_0 - a_1b_1)/b_0; \quad e_\sigma^2(a_0) = e_{\sigma_1}^2 \bullet b_2/D$$

$$a_1 = (b_0c_1 - b_1c_0)/D = (c_0 - a_0b_0)/b_1; \quad e_\sigma^2(a_1) = e_{\sigma_1}^2 \bullet b_0/D$$

$$D = b_0b_2 - b_1^2; \quad e_{\sigma_0}^2(s) = e_{\sigma_1}^2/b_0$$

where

$$\begin{aligned}(n-2)e_{\sigma_1}^2 &= d_0 - (a_0c_0 + a_1c_1) \\ &= S_C\end{aligned}$$

[Note N4.11]

Example 4.11c. If

$$F(x) = a_1x + a_2x^2$$

then, in Case A, with symbolism in (4.11-1),

$$\begin{aligned}a_1 &= (b_4c_1 - b_3c_2)/D = (c_1 - a_2b_3)/b_2; \quad e_{\sigma}^2(a_1) = e_{\sigma_1}^2 \bullet b_4/D \\ a_2 &= (b_2c_2 - b_3c_1)/D = (c_1 - a_1b_2)/b_3; \quad e_{\sigma}^2(a_2) = e_{\sigma_1}^2 \bullet b_2/D \\ D &= b_2b_4 - b_3^2; \quad n \bullet e_{\sigma_0}^2 = e_{\sigma_1}^2\end{aligned}$$

where

$$\begin{aligned}(n-2)e_{\sigma_1}^2 &= d_0 - (a_1c_1 + a_2c_2) \\ &= S_A\end{aligned}$$

[Note N4.11]

In Case B, with symbolism in (4.11-2),

$$\begin{aligned}a_1 &= (B_4C_1 - B_3C_2)/D = (C_1 - a_2B_3)/B_2; \quad e_{\sigma}^2(a_1) = E_{\sigma_1}^2 \bullet B_4/D \\ a_2 &= (B_2C_2 - B_3C_1)/D = (C_1 - a_1B_2)/B_3; \quad e_{\sigma}^2(a_2) = E_{\sigma_1}^2 \bullet B_2/D \\ D &= B_2B_4 - B_3^2; \quad n \bullet E_{\sigma_0}^2(s) = E_{\sigma_1}^2\end{aligned}$$

where

$$\begin{aligned}(n-2)E_{\sigma_1}^2 &= n - (a_1C_1 + a_2C_2) \\ &= S_B\end{aligned}$$

[Note N4.11]

In Case C, with symbolism in (4.11-3),

$$\begin{aligned}a_1 &= (b_4c_1 - b_3c_2)/D = (c_1 - a_2b_3)/b_2; \quad e_{\sigma}^2(a_1) = e_{\sigma_1}^2 \bullet b_4/D \\ a_2 &= (b_2c_2 - b_3c_1)/D = (c_1 - a_1b_2)/b_3; \quad e_{\sigma}^2(a_2) = e_{\sigma_1}^2 \bullet b_2/D \\ D &= b_2b_4 - b_3^2; \quad b_0 \bullet e_{\sigma_0}^2(s) = e_{\sigma_1}^2\end{aligned}$$

where

$$(n - 2)e_{\sigma_1}^2 = d_0 - (a_1c_1 + a_2c_2) \\ = S_C$$

[Note N4.11]

Example 4.11d. If

$$F(x) = a_0 + a_1x + a_2x^2$$

then, in Case A, a_0 , a_1 , and a_2 are obtained by solution of the three simultaneous equations

$$(4.11-4) \quad \begin{aligned} a_0b_0 + a_1b_1 + a_2b_2 &= c_0 \\ a_0b_1 + a_1b_2 + a_2b_3 &= c_1 \\ a_0b_2 + a_1b_3 + a_2b_4 &= c_2 \end{aligned}$$

Also,

$$ne_{\sigma_0}^2(s) = e_{\sigma_1}^2$$

where

$$(n - 3)e_{\sigma_1}^2 = d_0 - (a_0c_0 + a_1c_1 + a_2c_2) \\ = S_A$$

[Note N4.11]

In Case B, a_0 , a_1 , and a_2 are obtained by solution of three simultaneous equations like those of Eq. (4.11-4) except that the lowercase b_k 's and c_k 's are replaced by uppercase B_k 's and C_k 's, because the symbolism of (4.11-1) is replaced by the symbolism of (4.11-2). Also,

$$nE_{\sigma_0}^2(s) = E_{\sigma_1}^2$$

where

$$(n - 3)E_{\sigma_1}^2 = n - (a_0C_0 + a_1C_1 + a_2C_2) \\ = S_B$$

[Note N4.11]

In Case C, Eq. (4.11-4) holds, except that the b_k 's and c_k 's now have the meanings given in (4.11-3). Also,

$$b_0e_{\sigma_0}^2(s) = e_{\sigma_1}^2$$

where

$$(n-3)e_{\sigma_1}^2 = d_0 - (a_0c_0 + a_1c_1 + a_2c_2) \\ = S_C$$

[Note N4.11]

Example 4.11e. If n points (x_i, s_i) , $(i = 1, 2, \dots, n)$ are given, a cubic representation

$$F(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

is assumed, and it is required that the ordinate and slope at x_1 be Y_1 and Y'_1 , respectively, then, under the *a priori* assumption that all points have equal weight, the value of the coefficients which satisfy the least-squares conditions, and the associated rms deviations, are given by

$$a_3 = (b_1c_1 - b_2c_0) / (b_1b_3 - b_2^2)$$

$$a_2 = (c_0 - a_3b_2) / b_1$$

$$a_1 = Y'_1 - 2a_2x_1 - 3a_3x_1^2$$

$$a_0 = Y_1 - Y'_1x_1 + a_2x_1^2 + 2a_3x_1^3$$

$$(n-2)e_{\sigma_1}^2 = d_0 - a_2c_0 - a_3c_1$$

$$e_{\sigma_0}(s) = e_{\sigma_1} / \sqrt{n}$$

where the symbolism for this example is

$$b_1 = \sum \alpha_i^4; \quad b_2 = \sum \alpha_i^2 \beta_i; \quad b_3 = \sum \beta_i^2; \quad c_0 = \sum \alpha_i^2 \eta_i; \quad c_1 = \sum \beta_i \eta_i; \quad d_0 = \sum \eta_i^2$$

and

$$\alpha_i \equiv x_i - x_1; \quad \beta_i \equiv (x_i - x_1)(x_i + 2x_1); \quad \eta_i \equiv s_i - Y_1 - Y'_1 \alpha_i$$

This example is applicable to the piecewise fitting of contiguous segments in order to simulate a graphical construction by means of a French curve. In such an application, if the abscissae are more or less equally spaced, n is usually 4 and the x_1 of any segment is taken as the x_{n-1} of the preceding segment. Y_1 and Y'_1 of the segment are taken as the ordinate and slope of the least-squares cubic at x_{n-1} of the preceding segment.

4.12 Correlation between two variables when each is subject to error. If

$$(4.12-1) \quad y = a_0 + a_1x$$

and only the measurements y_i are likely to have significant error, the criterion $e_{\sigma_1}(y)$ is a reliable measure of how well Eq. (4.12-1) agrees with the measured data. However, if both x_i and y_i are likely to have significant error, a criterion that considers both uncertainties is the *correlation coefficient* r . Its formulation in terms of the quantities given in (4.11-3) is

$$(4.12-2) \quad r = \frac{b_0c_1 - b_1c_0}{[(b_0d_0 - c_0)^2(b_0b_2 - b_1^2)]^{1/2}}$$

An alternative formulation that clarifies the meaning of r is

$$(4.12-3) \quad r^2 = a_1A_1$$

where

a_1 is the slope dy/dx deduced when Eq. (4.12-1) is derived on the assumption that x_i is without error and each measurement y_i has weight w_i ;

A_1 is the slope dx/dy deduced when the equation

$$(4.12-4) \quad x = A_0 + A_1y$$

is derived on the assumption that y_i is without error and each measurement x_i has weight w_i . The formula for A_1 follows from Example 4.11b after symbols x_i and y_i are interchanged.

In Eq. (4.12-3), the positive sign of $\sqrt{r^2}$ is taken if a_1 is positive, and the negative sign if a_1 is negative. Equation (4.12-3) thus represents the ratio of the two values of dy/dx which are obtained when one first assumes y_i alone to have error and then assumes x_i alone to have error. [Note N4.12.1]

A common use of the correlation coefficient is to determine which of several variables is more likely to serve as a means of estimating the value of a related physical variable. The larger the absolute value of r , the better the correlation. The maximum possible value of $|r|$ is unity, which represents perfect correlation. However, comparisons among different values of r must always be made with an appreciation of the fact that a certain uncertainty exists in r , just as an uncertainty exists in a measure of dispersion like e_{σ_1} (Sec. 3.22). When there are n equally weighted pairs of data, and $|r| > 0.8$, the rms uncertainty $e_r(r)$ in r is given by

$$(4.12-5a) \quad e_r(r) \approx \beta \cdot (1 - r^2)(1 - r\beta)$$

where

$$(4.12-5b) \quad \beta = \pm 1/\sqrt{(n-3)}.$$

The positive and negative values of β lead to different magnitudes of $e_r(r)$. [Note N4.12.2, which refers to Ref. 4-1]

If the relation between x and y is strongly nonlinear and requires three or more constants for its description, the correlation coefficient must be replaced by an *index of correlation* ρ . This index is treated in Note N4.12.3.

4.13 Weight of transformed variables. If

$$(4.13-1) \quad y = f(x, a, b, \dots)$$

and the most probable values of a, b, \dots are to be determined from n pairs of measurements (x_i, y_i) , ($i = 1, 2, \dots, n$), a transformation of the dependent variable may be necessary to facilitate the determination, so that

$$(4.13-2) \quad s = s(y) = F(X, A, B, \dots)$$

where X, A, B, \dots may be the same as x, a, b, \dots or may be simple functions of x, a, b, \dots that involve no additional unknown parameters. When such a transformation is made, an appropriate change must be made in the weight assigned to each observation.

The weight of an observation is always inversely proportional to the square of the estimated uncertainty in the dependent variable. In the method of least squares, one seeks to minimize the sum of the appropriately weighted squares of the difference between the measurement and the value predicted by an assumed equation. Thus, for Eq. (4.13-1),

$$(4.13-3a) \quad \sum [f(x_i, a, b, \dots) - y_i]^2 w_i = \text{a minimum}$$

where

$$(4.13-3b) \quad w_i \equiv w(y_i) = \text{weight of } y_i.$$

(All summations are from $i = 1$ to $i = n$.) For the transformation (4.13-2), it is necessary that

$$(4.13-4a) \quad \sum [F(X_i, A, B, \dots) - s_i]^2 W_i = \text{a minimum}$$

where

$$(4.13-4b) \quad W_i \equiv W(s_i) = \text{weight of } s_i \\ = (\text{weight of } y_i) / (s'_i)^2$$

and

$$(4.13-5) \quad s'_i \equiv (ds/dy)_i.$$

The value of the minimum is a measure of how well the assumed function fits the data. In the case of Eq. (4.13-1), the quantities e_{o1} or E_{o1} given by Eq. (4.5-7) provide a measure of the fit. For future use of Eq. (4.13-1), the rms uncertainty in y is given by $e_{o0}(y)$ or $E_{o0}(y)$, where

$$(4.13-6a) \quad (n - m) (\sum w_i) e_{o0}^2(y) = \sum [f(x_i, a, b, \dots) - y_i]^2 w_i$$

or

$$(4.13-6b) \quad (n - m) E_{o0}^2(y) = \sum \{ [f(x_i, a, b, \dots) / y_i] - 1 \}^2$$

the latter equation applying to the case of equal *a priori* fractional uncertainty in the value of y_i .

If the data y_i fit $f(x_i, a, b, \dots)$ exactly, then $e_{o0}(y) = 0$ so that any choice of w_i will lead to the same value of the constants a, b, \dots . The values of $e_{o0}(y)$ and a, b, \dots become increasingly dependent on the choice of w_i as the scatter of the data increases.

Example 4.13a. If $s = \ln y$

$$\text{so that} \quad s' = 1/y$$

then

- (1) if, *a priori*, $\delta y = \text{a constant}$, the weight of s_i , by Eq. (4.13-4b), may be taken as

$$W_i = 1 / (1/y_i)^2 = y_i^2;$$

(2) if, *a priori*, $\delta y/y = a$ constant, then, since

$$\delta y = y \times (\text{a constant}) ;$$

the weight of y_i is $1/y_i^2$ and the weight of s_i is, by Eq. (4.13-4b),

$$W_i = (1/y_i)^2 / (1/y_i)^2 = 1 ;$$

(3) if, *a priori*, y_i has weight w_i , then

$$W_i = w_i y_i^2$$

Example 4.13b. If $s = \log y$

so that $s' = (1/y)/\ln 10$

then, since $\ln 10$ is merely a constant, the values of W_i in Example 4.13a also may be used for this example.

Example 4.13c. If $y = e^{ax^2}$

and the most probable value of a is to be determined from a number of measurements (x_i, y_i) on the *a priori* assumption that δy is constant, then Example 4.13a leads to

$$\sum (ax^2 - \ln y)^2 y^2 = \text{a minimum}$$

so that

$$a = \frac{\sum (x^2 y^2 \cdot \ln y)}{\sum (x^4 y^2)}$$

$$(n - 1)e_{\sigma_1}^2 = \sum (y \cdot \ln y)^2 - a \sum (x^2 y^2 \cdot \ln y)$$

$$n \cdot e_{\sigma_0}^2(y) = e_{\sigma_1}^2$$

Other examples appear in Sec. 4.14.

4.14 Examples of least-squares solutions using transformed variables. In this section, some examples will be given of least-squares solutions for functions $y = f(x)$ where, for each pair of observations (x_i, y_i) , ($i = 1, 2, \dots, n$), the dependent measurement y_i is assumed *a priori*, to have a weight w_i . Two special cases are readily derived from this more general case:

- A. If all measurements of y_i are presumed to have the same uncertainty, then one may set all w_i equal to unity, noting that, thereby, $\sum w_i = n$. (All summations are from $i = 1$ to $i = n$.)
- B. If all measurements of y_i are presumed to have the same fractional uncertainty, then one sets $w_i = 1/y_i^2$, noting that, in this case, $\sum w_i y_i^2 = n$.

Group I. Two undetermined constants. Linear relations. In this group of examples, the relation

$$(4.14-1) \quad y = f(x, a, b)$$

will be transformed into the linear relation

$$(4.14-2) \quad s = A + BX$$

and the residuals

TABLE 4.14.1.—TERMS IN LEAST-SQUARES SOLUTIONS FOR SOME FUNCTIONS WITH TWO UNDETERMINED CONSTANTS

Example: Formula:	(a) $y = ax^b$	(b) $x = ay^b$	(c) $y = a\alpha^{bx}$	(d) $x = a\alpha^{by}$	(e) $y = a\alpha^{b/x}$	(f) $x = a\alpha^{b/y}$	(g) $y = a/(1+bx)$	(h) $x = a/(1+by)$
a	ϵ^A	$\epsilon^{-A/B}$	ϵ^A	$\epsilon^{-A/B}$	ϵ^A	$\epsilon^{-A/B}$	$1/A$	$-B/A$
b	B	$1/B$	$B/\ln \alpha$	$1/(B \ln \alpha)$	$B/\ln \alpha$	$1/(B \ln \alpha)$	B/A	$-1/A$
b_0	Σwy^2	Σwy^2	Σwy^2	Σw	Σwy^2	Σwy^4	Σwy^4	Σw
b_1	$\Sigma wy^2 \ln x$	$\Sigma wy^2 \ln x$	Σwx^2	$\Sigma w \ln x$	$\Sigma wy^2/x$	$\Sigma wy^4 \ln x$	Σwxy^4	$\Sigma w/x$
b_2	$\Sigma wy^2 \ln^2 x$	$\Sigma wy^2 \ln^2 x$	$\Sigma wx^2 y^2$	$\Sigma w \ln^2 x$	$\Sigma wy^2/x^2$	$\Sigma wy^4 \ln^2 x$	$\Sigma wx^2 y^4$	$\Sigma w/x^2$
c_0	$\Sigma wy^2 \ln y$	$\Sigma wy^2 \ln y$	$\Sigma wy^2 \ln y$	Σwy	$\Sigma wy^2 \ln y$	Σwy^3	Σwy^3	Σwy
c_1	$\Sigma wy^2 \ln x \ln y$	$\Sigma wy^2 \ln x \ln y$	$\Sigma wx^2 \ln y$	$\Sigma wy \ln x$	$\Sigma (wy^2 \ln y)/x$	$\Sigma wy^3 \ln x$	Σwxy^3	$\Sigma wy/x$
d_0	$\Sigma wy^2 \ln^2 y$	$\Sigma wy^2 \ln^2 y$	$\Sigma wy^2 \ln^2 y$	Σwy^2	$\Sigma wy^2 \ln^2 y$	Σwy^2	Σwy^2	Σwy^2
s	$\ln y$	$\ln y$	$\ln y$	y	$\ln y$	$1/y$	$1/y$	y
X	$\ln x$	$\ln x$	x	$\ln x$	$1/x$	$\ln x$	x	$1/x$
W	wy^2	wy^2	wy^2	w	wy^2	wy^4	wy^4	w
$e_0^2(a)/e_{01}^2(s)$	$a^2 b_2/D$	$\approx a^2 d_0/(B^4 D)$	$a^2 b_2/D$	$\approx a^2 d_0/(B^4 D)$	$a^2 b_2/D$	$\approx a^2 d_0/(B^4 D)$	$b_2/(A^4 D)$	$\approx d_0/(A^4 D)$
$e_0^2(b)/e_{01}^2(s)$	b_0/D	$b_0/(B^4 D)$	$b_0/(D \ln^2 \alpha)$	$b_0/(B^4 D \ln^2 \alpha)$	$b_0/(D \ln^2 \alpha)$	$b_0/(B^4 D \ln^2 \alpha)$	$\approx d_0/(A^4 D)$	$b_2/(A^4 D)$

Procedure: Given a set of (x_i, y_i, w_i) ,

1. calculate $b_0, b_1, b_2, c_0, c_1, d_0$ from this table;
2. calculate A, B, D from Eqs. (4.14-4) and (4.14-5);
3. calculate a, b from this table.

If desired,

4. calculate $e_{01}(s)$ from Eq. (N4.14-5) in Note N4.14.1;
5. calculate $e_0(a), e_0(b)$ from this table.

TABLE 4.14.2.—TERMS IN LEAST-SQUARES SOLUTIONS FOR SOME FUNCTIONS WITH THREE UNDETERMINED CONSTANTS

Example:	(a)	(b)	(c)
Formula:	$y = a\alpha^{bx} + cx^2$	$y = a\alpha^{(b/x) + (c/x^2)}$	$y = a/(1 + bx + cx^2)$
a	ϵ^A	ϵ^A	$1/A$
b	$B/\ln \alpha$	$B/\ln \alpha$	B/A
c	$C/\ln \alpha$	$C/\ln \alpha$	C/A
$b_j (j = 0 \text{ to } 4)$	$\Sigma wx^j y^2$	$\Sigma w\gamma^{2/x^j}$	$\Sigma wx^j y^4$
$c_j (j = 0 \text{ to } 2)$	$\Sigma wx^j y^2 \ln y$	$\Sigma (w\gamma^{2/x^j} \ln y)/x^j$	$\Sigma wx^j y^3$
d_0	$\Sigma w\gamma^{2/x^2} \ln^2 y$	$\Sigma w\gamma^{2/x^2} \ln^2 y$	$\Sigma w\gamma^2$
s	$\ln y$	$\ln y$	$1/y$
X	x	$1/x$	x
W	$w\gamma^2$	$w\gamma^2$	$w\gamma^4$
$e_a^2(a)/e_{\sigma_1}^2(s)$	$a^2(b_2 b_4 - b_3^2)/D$	$a^2(b_2 b_4 - b_3^2)/D$	$(b_2 b_4 - b_3^2)/(DA^4)$
$e_b^2(b)/e_{\sigma_1}^2(s)$	$(b_0 b_4 - b_2^2)/(D \ln^2 \alpha)$	$(b_0 b_4 - b_2^2)/(D \ln^2 \alpha)$	$= (b_4 d_0 - c_2^2)/(DA^4)$
$e_c^2(c)/e_{\sigma_1}^2(s)$	$(b_0 b_2 - b_1^2)/(D \ln^2 \alpha)$	$(b_0 b_2 - b_1^2)/(D \ln^2 \alpha)$	$= (b_2 d_0 - c_1^2)/(DA^4)$

Procedure: Given a set of (x_i, y_i, w_i) ,

1. calculate the b_j 's, c_j 's, and d_0 from this table;
2. calculate A, B, C by solving Eqs. (4.14-10);
3. calculate a, b, c from this table.

If desired,

4. calculate $e_{\sigma_1}(s)$ from Eq. (N4.14-11) in Note N4.14.2;
5. calculate $e_{\sigma}(a), e_{\sigma}(b), e_{\sigma}(c)$ from this table. (D is the 3rd-order determinant of the coefficients (the b_j 's) of (4.14-10).)

$$(4.14-3) \quad v_{s_i} \equiv v(s_i) = A + BX_i - s_i$$

will be assigned appropriate weights W_i in accordance with Sec. 4.13.

The constants A and B are given by

$$(4.14-4a) \quad A = (b_2 c_0 - b_1 c_1)/D$$

$$(4.14-4b) \quad B = (b_0 c_1 - b_1 c_0)/D$$

where

$$(4.14-5) \quad D = b_0 b_2 - b_1^2$$

and

$$(4.14-6) \quad b_0 = \Sigma W_i; \quad b_1 = \Sigma W_i X_i; \quad b_2 = \Sigma W_i X_i^2; \quad c_0 = \Sigma W_i s_i; \quad c_1 = \Sigma W_i X_i s_i$$

Explicit formulations of these summations for each example are listed in Table 4.14.1, which also lists expressions for a and b and, as a matter of incidental interest, the transformed variables $s, W,$ and X . [Note N4.14.1] For ease of tabulation, the index subscript i has been omitted from $x, y,$ and w in the summations.

The solution of this set yields the $(m+1)$ constants a_k ($k = 0, 1, \dots, m$). [Note N4.15.2]
 The uncertainty $e_{\sigma 0}(y)$ is given by

$$(4.15-4) \quad e_{\sigma 0}^2(y) = e_{\sigma 1}^2/b_0; \quad e_{\sigma 1} \equiv e_{\sigma 1}(y)$$

where

$$(4.15-5a) \quad (n-m-1)e_{\sigma 1}^2 = d_0 - \sum_0^m a_k c_k$$

$$(4.15-5b) \quad = \sum_0^n w_i v_i^2$$

[Note N4.11]

Example 4.15. A quadratic parabola

$$y = a_0 + a_1x + a_2x^2$$

is to be fitted to the four points $(x_i, y_i) = (0, 1), (1, 0.368), (2, 0.135), (3, 0.050)$ in the interval $0 \leq x \leq 3$. Then

A. If each measurement y_i is presumed to have equal uncertainty, so that $w_i = 1$, the solution is

$$y = 0.988 - 0.719x + 0.1368x^2 \pm 0.03.$$

B. If each measurement y_i is presumed to have equal fractional uncertainty, so that $w_i = 1/y_i^2$, the solution is

$$y = (0.895 - 0.585x + 0.1013x^2) (1 \pm 0.08).$$

The numbers following the \pm sign represent $e_{\sigma 0}(y)$ in case A and $E_{\sigma 0}(y)$ in case B. These two parabolas happen to be approximations to the curve

$$y = \exp(-x)$$

in the x -interval $0 < x < 3$. They are not useful outside that interval.

4.16 Least-squares solution for the general case of linearly related undetermined constants. Assume that

$$(4.16-1) \quad s(x) = \sum_{k=1}^m a_k f_k(x)$$

where each f_k is a function of x that contains no undetermined constants, that there are n pairs of measurements (x_i, s_i) ($i = 1, 2, \dots, n$), and that the i th measurement has weight w_i . (If all measurements are presumed to have equal uncertainty, set $w_i = 1$ and note that $\Sigma w_i = n$; if all measurements are presumed to have equal fractional uncertainty, set $w_i = 1/s_i^2$ and note that $\Sigma w_i s_i^2 = n$.) All summations not otherwise identified will be from $i = 1$ to $i = n$. Let

$$(4.16-2) \quad f_{ki} \equiv f_k(x_i)$$

$$s = a_1 + a_2x + a_3T_2 + a_4T_3$$

is used to describe the temperature s measured by a W5Re/W26Re thermocouple, in the temperature range $1000 \leq s \leq 2200$ K, where x is the nondimensional representation

$$x = (2X - X_1 - X_2)/(X_2 - X_1)$$

and X = emf, in mV, generated when the hot junction is at s kelvin and the cold junction is at 300 kelvin.

$$X_1 = \text{value of } X \text{ when } s = 1000 \text{ K}$$

$$X_2 = \text{value of } X \text{ when } s = 2200 \text{ K.}$$

The calibration data used by the manufacturer in 1974 lead to the five points (x_i, s_i)

$$(-1, 1000); (-0.5, 1260); (0, 1540); (0.5, 1840); (1, 2200).$$

If each of these points has equal weight, the values

$$a_1 = 1568; a_2 = 596; a_3 = 32; a_4 = 4$$

will yield future measurements of s with an uncertainty

$$e_{s0}(s) = 0.7 \text{ K}$$

if there is no change in thermocouple characteristics.

Example 4.16b. If there are n sets of observations (x_i, y_i, s_i) , where $i = 1, 2, \dots, n$, and one assumes that

$$(4.16-9) \quad s = a_0 + a_1x + a_2y$$

and that each observation s_i has weight w_i , then the constants a_0, a_1, a_2 are obtained by solution of the simultaneous equations

$$b_{00}a_0 + b_{10}a_1 + b_{20}a_2 = c_{00}$$

$$(4.16-10) \quad b_{10}a_0 + b_{20}a_1 + b_{11}a_2 = c_{10}$$

$$b_{01}a_0 + b_{11}a_1 + b_{02}a_2 = c_{01}$$

where

$$(4.16-11) \quad b_{jk} = \sum w_i x_i^j y_i^k, \quad c_{jk} = \sum w_i s_i x_i^j y_i^k$$

and all summations are from $i = 1$ to $i = n$. The uncertainty $e_{s0}(s)$ to be expected in future use of Eq. (4.16-9) is given by

$$(4.16-12a) \quad e_{s0}^2(s) = e_{s1}^2/b_{00}$$

where

$$(4.16-12b) \quad (n-3) e_{\sigma_1}^2 = d_0 - (a_0 c_{00} + a_1 c_{10} + a_2 c_{01})$$

$$(4.16-12c) \quad = \sum w_i v_i^2$$

[Note N4.11]

$$(4.16-12d) \quad d_0 = \sum w_i s_i^2$$

$$(4.16-12e) \quad v_i = s(x_i, y_i) - s_i$$

If all measurements are presumed to be of equal weight, then w_i may be set equal to unity in (4.16-11), noting that $b_{00} = n$.

If all measurements are presumed to have the same fractional uncertainty, then (4.16-11) may be replaced by

$$(4.16-13) \quad b_{jk} = \sum x_i^j y_i^k / s_i^2, \quad c_{jk} = \sum x_i^j y_i^k / s_i$$

and (4.16-12) may be replaced by

$$(4.16-14a) \quad E_{\sigma_0}^2(s) = E_{\sigma_1}^2 / b_{00}$$

$$(4.16-14b) \quad (n-3) E_{\sigma_1}^2 = n - (a_0 c_{00} + a_1 c_{10} + a_2 c_{01})$$

$$(4.16-14c) \quad = \sum V_i^2$$

$$(4.16-14d) \quad V_i = [s(x_i, y_i) / s_i] - 1$$

[Note N4.11]

Example 4.16c. If there are n sets of observations (x_i, y_i, s_i) , where

$i = 1, 2, \dots, n$, and one assumes that

$$(4.16-15) \quad s = a_0 + a_1 x + a_2 y + a_3 xy$$

with each observation s_i having weight w_i , then the constants a_0, a_1, a_2, a_3 are obtained by solution of the simultaneous equations

$$(4.16-16) \quad \begin{aligned} b_{00}a_0 + b_{10}a_1 + b_{01}a_2 + b_{11}a_3 &= c_{00} \\ b_{10}a_0 + b_{20}a_1 + b_{11}a_2 + b_{21}a_3 &= c_{10} \\ b_{01}a_0 + b_{11}a_1 + b_{02}a_2 + b_{12}a_3 &= c_{01} \\ b_{11}a_0 + b_{21}a_1 + b_{12}a_2 + b_{22}a_3 &= c_{11} \end{aligned}$$

where

$$(4.16-17) \quad b_{jk} = \sum w_i x_i^j y_i^k, \quad c_{jk} = \sum w_i s_i x_i^j y_i^k$$

and all summations are from $i = 1$ to $i = n$.

The uncertainty $e_{s0}(s)$ to be expected in future use of Eq. (4.16-15) is given by

$$(4.16-18a) \quad e_{s0}^2(s) = e_{s1}^2/b_{00}$$

where

$$(4.16-18b) \quad (n-4) e_{s1}^2 = d_0 - (a_0c_{00} + a_1c_{10} + a_2c_{01} + a_3c_{11})$$

$$(4.16-18c) \quad = \sum w_i v_i^2$$

$$(4.16-18d) \quad d_0 = \sum w_i s_i^2$$

[Note N4.11]

$$(4.16-18e) \quad v_i = s(x_i, y_i) - s_i$$

If all measurements are presumed to have the same weight, w_i may be set equal to unity in (4.16-17), noting that $b_{00} = n$.

If all measurements are presumed to have the same fractional uncertainty, then (4.16-17) may be replaced by

$$(4.16-19) \quad b_{jk} = \sum x_i^j y_i^k / s_i^2, \quad c_{jk} = \sum x_i^j y_i^k / s_i$$

and (4.16-12) may be replaced by

$$(4.16-20a) \quad E_{s0}^2(s) = E_{s1}^2 / b_{00}$$

$$(4.16-20b) \quad (n-4) E_{s1}^2 = n - (a_0c_{00} + a_1c_{10} + a_2c_{01} + a_3c_{11})$$

$$(4.16-20c) \quad = \sum V_i^2$$

[Note N4.11]

$$(4.16-20d) \quad V_i = [s(x_i, y_i) / s_i] - 1$$

Note: This procedure is often convenient for the determination of the values a , b , c in the expression

$$(4.16-21) \quad s = a(1 + bx)(1 + cy)$$

although these values are not rigorously the most probable values, because the expression (4.16-21) contains fewer undetermined constants than the expression (4.16-15). The procedure is adequate if $a_1 a_2 / (a_3 a_0)$ is sufficiently close to unity to be acceptable for the purposes of the approximation. Then $a = a_0$, $b = a_1 / a_0$; $c = a_2 / a_0$.

Equation (4.16-21) is often a convenient form of representing the dependence of s on the variables x and y , when the dependence is slight, and when s appears to vary linearly with each of the variables when the other is held constant. For example, s may represent the density of a liquid and x and y may represent liquid temperature and pressure.

Example 4.16d. If there are n sets of observations (x_i, y_i, s_i) , where $i = 1, 2, \dots, n$ and one assumes that

$$(4.16-22) \quad s = a_0 + a_1 x + a_2 x^2 + a_3 y + a_4 y^2$$

with each observation s_i having weight w_i , then the constants a_0, a_1, a_2, a_3, a_4 are obtained by solution of the simultaneous equations

$$\begin{aligned}
 & b_{00}a_0 + b_{10}a_1 + b_{20}a_2 + b_{01}a_3 + b_{02}a_4 = c_{00} \\
 & b_{10}a_0 + b_{20}a_1 + b_{30}a_2 + b_{11}a_3 + b_{12}a_4 = c_{10} \\
 (4.16-23) \quad & b_{20}a_0 + b_{30}a_1 + b_{40}a_2 + b_{21}a_3 + b_{22}a_4 = c_{20} \\
 & b_{01}a_0 + b_{11}a_1 + b_{21}a_2 + b_{02}a_3 + b_{03}a_4 = c_{01} \\
 & b_{02}a_0 + b_{12}a_1 + b_{22}a_2 + b_{03}a_3 + b_{04}a_4 = c_{02}
 \end{aligned}$$

where

$$(4.16-24) \quad b_{jk} = \sum w_i x_i^j y_i^k, \quad c_{jk} = \sum w_i s_i x_i^j y_i^k$$

and all summations are from $i = 1$ to $i = n$.

The uncertainty $e_{\sigma 0}(s)$ to be expected in future use of Eq. (4.16-22) is given by

$$(4.16-25a) \quad e_{\sigma 0}^2(s) = e_{\sigma 1}^2/b_{00}$$

where

$$\begin{aligned}
 (4.16-25b) \quad (n-5) e_{\sigma 1}^2 &= d_0 - (a_0 c_{00} + a_1 c_{10} + a_2 c_{20} + a_3 c_{01} + a_4 c_{02}) \\
 &= \sum w_i v_i^2
 \end{aligned}$$

[Note N4.11]

$$(4.16-25c) \quad d_0 = \sum w_i s_i^2$$

$$(4.16-25d) \quad v_i = s(x_i, y_i) - s_i.$$

If all measurements are presumed to have equal weight, w_i may be set equal to unity, noting that $b_{00} = n$.

If all measurements are presumed to have equal fractional uncertainty, then (4.16-24) may be replaced by

$$(4.16-26) \quad b_{jk} = \sum x_i^j y_i^k / s_i^2, \quad c_{jk} = \sum x_i^j y_i^k / s_i$$

and (4.16-25) may be replaced by

$$(4.16-27a) \quad E_{\sigma 0}^2(s) = E_{\sigma 1}^2/b_{00}$$

where

$$(4.16-27b) \quad (n-5) E_{\sigma 1}^2 = n - (a_0 c_{00} + a_1 c_{10} + a_2 c_{20} + a_3 c_{01} + a_4 c_{02})$$

$$(4.16-27c) \quad = \sum V_i^2$$

[Note N4.11]

$$(4.16-27d) \quad V_i = [s(x_i, y_i) / s_i] - 1$$

Example 4.16e. If there are n sets of observations (x_i, y_i, s_i) ,

where $i = 1, 2, \dots, n$, and one assumes that

$$(4.16-28) \quad s = y \bullet (a_0 + a_1 x + a_2 x^2 + a_3 y + a_4 xy)$$

with each observation of s having the same weight, then the constants a_0 to a_4 are obtained by solution of the five simultaneous equations

$$(4.16-29) \quad \begin{aligned} a_0 b_{02} + a_1 b_{12} + a_2 b_{22} + a_3 b_{03} + a_4 b_{13} &= c_{01} \\ a_0 b_{12} + a_1 b_{22} + a_2 b_{32} + a_3 b_{13} + a_4 b_{23} &= c_{11} \\ a_0 b_{22} + a_1 b_{32} + a_2 b_{42} + a_3 b_{23} + a_4 b_{33} &= c_{21} \\ a_0 b_{03} + a_1 b_{13} + a_2 b_{23} + a_3 b_{04} + a_4 b_{14} &= c_{02} \\ a_0 b_{13} + a_1 b_{23} + a_2 b_{33} + a_3 b_{14} + a_4 b_{24} &= c_{12} \end{aligned}$$

where

$$(4.16-30) \quad b_{jk} = \sum x_i^j y_i^k, \quad c_{jk} = \sum s_i x_i^j y_i^k$$

and all summations are from 0 to n .

The uncertainty $e_{s0}(s)$ to be expected in future use of Eq. (4.16-28) is given by

$$(4.16-31a) \quad e_{s0}^2(s) = e_{s1}^2/n$$

$$(4.16-31b) \quad (n-5)e_{s1}^2 = d_0 - (a_0 c_{01} + a_1 c_{11} + a_2 c_{21} + a_3 c_{02} + a_4 c_{12})$$

$$(4.16-31c) \quad d_0 = \sum s_i^2$$

Example 4.16f. If there are n observations (x_i, y_i, s_i) , where $i = 1, 2, \dots, n$, and one assumes that

$$(4.16-32) \quad s = x^{a+b} y^c$$

with each observation of s having *a priori*, the same fractional error, then the constants a, b, c are obtained by solution of the three simultaneous equations

$$(4.16-33) \quad \begin{aligned} a \bullet b_{020} + b \bullet b_{120} + c \bullet b_{011} &= c_{010} \\ a \bullet b_{120} + b \bullet b_{220} + c \bullet b_{111} &= c_{110} \\ a \bullet b_{011} + b \bullet b_{111} + c \bullet b_{002} &= c_{001} \end{aligned}$$

where

$$(4.16-34) \quad b_{jkl} = \sum x_i^j (\ln x_i)^k (\ln y_i)^l$$

$$c_{jkl} = \sum (\ln s_i) x_i^j (\ln x_i)^k (\ln y_i)^l$$

and all summations are from $i = 1$ to $i = n$.

The fractional uncertainty $E_{\sigma_0}(s)$ to be expected in future use of Eq. (4.16-32) is given by

$$(4.16-35a) \quad E_{\sigma_0}^2(s) = E_{\sigma_1}^2/n$$

$$(4.16-35b) \quad (n-3)E_{\sigma_1}^2 = d_0 - (a \bullet c_{010} + b \bullet c_{110} + c \bullet c_{001})$$

$$(4.16-35c) \quad d_0 = \sum (\ln s_i)^2$$

4.17 Least-squares solution for a first-order differential equation. Many instrument applications require the determination of the constants Y and τ of the differential equation

$$(4.17-1) \quad \tau(dy/dx) + y = Y$$

where y represents instrument indication, and x usually represents time or distance. In such applications, the values of y and x are usually known accurately, but the relation (4.17-1) may be only approximate, so that the most probable values of τ and Y are of interest; sometimes, only one of these is of interest. The two constants may be determined conveniently if the following conditions are met:

1. Measurements y_i are available at equal increments Δx of the independent variable x , so that

$$(4.17-2) \quad x_i = (i-1)\Delta x \quad (i = 1, 2, \dots).$$

The initial measurement, y_1 , will be considered to have been made at $x = 0$, so that $x_1 = 0$ and

$$(4.17-3) \quad y_i \equiv y(x_i)$$

2. Successive increments in y , described for brevity as

$$(4.17-4) \quad \Delta_i \equiv y_{i+1} - y_i \quad (i = 1, 2, \dots)$$

are all of the same sign and are sufficiently large and sufficiently accurate so that the sequence of Δ_i 's is monotonic; i.e., the absolute magnitude of each successive Δ_i is consistently smaller than or equal to the absolute magnitude of the preceding Δ_i .

3. All measurements are, *a priori*, deemed to be of equal weight.

Under these conditions, if the last observation is y_{n+1} , so that there are n values of Δ_i ,

$$(4.17-5) \quad Y = (b_2 c_0 - b_1 c_1) / D$$

where

$$(4.17-6) \quad D = nb_2 - b_1^2;$$

and

$$(4.17-7a) \quad \tau/\Delta x = -1/\ln [1 - (1/p)]$$

where

$$(4.17-7b) \quad p = (b_1c_0 - nc_1)/D. \quad [\text{Note N4.17}]$$

Here,

$$b_1 = \sum \Delta_i = y_{n+1} - y_1$$

$$b_2 = \sum \Delta_i^2 = 2 \sum (y_i^2 - y_i y_{i+1}) + y_{n+1}^2 - y_1^2$$

$$(4.17-8) \quad c_0 = \sum y_i$$

$$c_1 = \sum y_i \Delta_i = \sum (y_i y_{i+1} - y_i^2)$$

and all summations are from $i = 1$ to $i = n$.

In an application where many values of (x_i, y_i) are available but either τ or Y is suspected of changing systematically, though slightly, from the beginning to the end of the sequence of data taking, it is often helpful to apply Eqs. (4.17-5) to (4.17-8) only to the readings (four or more) in the vicinity of the region of principal interest. Excessively fine subdivision of this region is often undesirable because it detracts from the advantage of having larger and, hence, more accurate values of the increments Δ_i . In the presence of undesired high-frequency fluctuations in y_i , the use of larger subdivisions is often an effective means of satisfying the requirement for a monotonic sequence of Δ_i 's.

An estimate of the efficacy with which Eq. (4.17-1) fits the data is given by the quantity e_σ ,

where

$$(4.17-9a) \quad (n-2) e_\sigma^2 = d_0 - c_0 Y + c_1 p$$

$$(4.17-9b) \quad d_0 = \sum y_i^2$$

However, e_σ does not represent the rms value of the residual

$$(4.17-10) \quad v_i = [Y + (y_1 - Y)\epsilon^{-(i-1)\Delta x/\tau}] - y_i.$$

This rms value is usually smaller than e_σ , but less convenient to calculate.

If condition 1, that readings be at equal increments, is not fully met, one may choose an increment Δx that is near the average of the actual increments in the available data and then use interpolation to derive values of y_i at abscissae $(i-1)\Delta x$. A quadratic interpolation formula through four or five points closest to $(i-1)\Delta x$ usually suffices. Such a formula is provided by Example 4.11d, Case A, or by Sec. 4.19.

If condition 2, that the sequence of Δ_i 's be monotonic, is not fully met, then smoothing by the method of Sec. 4.18 may permit meeting the required condition.³ If the original Δ_i 's had only two significant figures and were of low accuracy, the use of smoothed values to an additional figure usually improves the accuracy of computation of Y and τ .

³In electrical circuits that will use an analog-to-digital converter to produce numerical data, prior smoothing by use of a low-pass filter may be more convenient.

Example 4.17a. An illustration of the effects of compromising between n and Δ_i is given by presuming some measured ordinates of the equation

$$y + dy/dx = 2$$

(where $\tau = 1$, $Y = 2$).

Case 1. If $\Delta x = 0.1$ and $y_i = 2.202, 2.183, 2.165, 2.150, 2.135, 2.122, 2.111, 2.100$, then $n = 7$, and $Y = 2.003$; $\tau/\Delta x = 9.75$; $e_\sigma = 0.007$.

Case 2. If $\Delta x = 0.2$ and $y_i = 2.183, 2.150, 2.122, 2.100$, then $n = 3$, and $Y = 1.999$; $\tau/\Delta x = 5.00$; $e_\sigma = 0.004$.

Case 3. If $\Delta x = 0.2$ and $y_i = 2.202, 2.165, 2.135, 2.111$, then $n = 3$, and $Y = 2.011$; $\tau/\Delta x = 4.64$; $e_\sigma = 0.001$.

Example 4.17b. An illustration of the effects of rounding is given by assuming some measured ordinates of the equation

$$y + dy/dx = 0$$

(where $\tau = 1$, $Y = 0$).

Case 1. If $\Delta x = 1$ and $y_i = 0.905, 0.333, 0.122, 0.045$, then $n = 3$, and $Y = -0.000_3$; $\tau/\Delta x = 1.000_2$; $e_\sigma = 0.001$.

Case 2. If $\Delta x = 1$ and $y_i = 0.90, 0.33, 0.12, 0.05$, then $n = 3$, and $Y = 0.007$; $\tau/\Delta x = 0.981$; $e_\sigma = 0.007$.

Example 4.17c. An illustration of the effects of smoothing is provided by assuming that the values of y_i in Example 4.17a, Case 1 are $y_i = 2.203, 2.182, 2.166, 2.149, 2.136, 2.121, 2.112, 2.099$. Then the Δ_i , which range from 0.009 to 0.021, do not form a monotonic sequence. Smoothing by the methods of Sec. 4.18, and use of an additional figure in the smoothed values yield the sequence $y_i = 2.2027, 2.1828, 2.1651, 2.1499, 2.1348, 2.1225, 2.1106, 2.0995$. Then

$$Y = 2.004; \tau/\Delta x = 9.56; e_\sigma = 0.006.$$

4.18 Smoothing of data. For some mathematical operations on empirical data, it is necessary that the deviations of the data from a smooth curve be small enough so that the equivalent of a first or second derivative may be obtainable. This result may sometimes be accomplished by fitting an appropriate equation to the data. This equation may be quite complex and may involve many empirical constants. In such case, better accuracy and greater convenience may sometimes be attained by piecewise fitting of a simple quadratic or cubic parabola to small groups of adjacent points, overlapping the groups to maintain continuity of the derivative. The analytical operation is strictly analogous to progressive use of French curves to draw a smooth curve through a large sequence of plotted points.⁴ In fact, the graphical operation is often more effective and convenient than the analytical one—given a graph on a scale large enough to reveal the least significant digit, one may then read the ordinates of the smooth curve at convenient abscissae.

When unfamiliar data are first acquired, a preliminary graphical construction may also help to reveal potential problems, like anomalous points, and to guide the smoothing operation.

Smoothing becomes particularly convenient when data are obtained at constant increments Δx of the independent variable. Let the sequence of data be represented by y_i ($i = 1, 2, \dots, k, \dots, n$). Let the smoothed value of y_i be \bar{y}_i . Then

⁴The operation resembles the "cubic spline" method of piecewise construction of a complex curve, but differs from it in the following important respect. The cubic spline method creates a curve that treats every point as perfectly accurate and, hence, passes through each point; the French-curve method or its analytical equivalent (Example 4.11e) provides smoothing that allows for the imperfect accuracy of individual points.

(4.18-1) $\bar{y}_1 = (19y_1 + 3y_2 - 3y_3 + y_4)/20$

(4.18-2) $\bar{y}_2 = (9y_1 + 13y_2 + 12y_3 + 6y_4 - 5y_5)/35$

(4.18-3) $\bar{y}_k = [12(y_{k-1} + y_{k+1}) + 17y_k - 3(y_{k-2} + y_{k+2})]/35 \quad 3 \leq k \leq (n - 2)$

(4.18-4) $\bar{y}_{n-1} = (9y_n + 13y_{n-1} + 12y_{n-2} + 6y_{n-3} - 5y_{n-4})/35$

(4.18-5) $\bar{y}_n = (19y_n + 3y_{n-1} - 3y_{n-2} + y_{n-3})/20$

Each of these equations represents the result of fitting a quadratic parabola to the four or five points specified in that equation.

This smoothing technique is applicable to empirical data presented in tabular form at constant increments of the independent variable. However, such smoothing may be facilitated by applying it to the tabulation of some function of y (such as the reciprocal, power, logarithm, or sine) rather than to y itself.⁵

4.19 Interpolation of data. If n pairs of data (x_i, y_i) ($i = 1, 2, \dots, n$) are available and a value of $y(x)$ is desired, where x is none of the x_i , a polynomial of the form

(4.19-1) $y = \sum a_k x^k \quad (k = 0, 1, \dots, n - 1)$

may be fitted to the data. The procedure is to form the n equations

$$a_0 b_0 + a_1 b_1 + \dots + a_k b_k + \dots + a_{n-1} b_{n-1} = c_0$$

$$a_0 b_1 + a_1 b_2 + \dots + a_k b_{k+1} + \dots + a_{n-1} b_n = c_1$$

(4.19-2)

$$a_0 b_{n-1} + a_1 b_n + \dots + a_k b_{k+n-1} + \dots + a_{n-1} b_{2n-2} = c_{n-1}$$

where

$$b_j = \sum x_i^j; \quad c_j = \sum x_i^j y_i; \quad (j = 0, 1, 2, \dots, n - 1)$$

and all summations are from $i = 1$ to $i = n$. (Note that $b_0 = n$ and $c_0 = \sum y_i$.) The equations are then solved for the a_k 's. The procedure is used most often with $n \leq 5$, because it is most effective when a few points in the vicinity of x are used to find the interpolated value $y(x)$.

An alternative formulation of the above procedure is Lagrange's formula:

(4.19-3a) $y(x) = P \cdot \sum [y_i / P_i]$

where

(4.19-3b) $P = \prod_i (x - x_i) \quad (i = 1, 2, \dots, n)$

(4.19-3c) $P_i = (x - x_i) \prod_j (x_i - x_j) \quad [j = 1, 2, \dots, (i - 1), (i + 1), \dots, n]$

⁵Smoothing, interpolation, and some other operations treated in this chapter are considered thoroughly and authoritatively in Ref. 4-2.

Equation (4.19-3a) is the same polynomial as Eq. (4.19-1), but yields only a numerical result, not an algebraic formula.

Example 4.19. Given $(x_i, y_i) = (1, 9), (2, 18), (3, 31)$ to find $y(2.5)$. Then $b_0 = 3, b_1 = 6, b_2 = 14, b_3 = 36, b_4 = 98, c_0 = 58, c_1 = 138, c_2 = 360$

and the solution of (4.19-2) leads to

$$y = 4 + 3x + 2x^2$$

so that

$$y(2.5) = 24.$$

Eq. (4.19-3) leads to

$$P = -3/8, P_1 = 1.5(-1)(-2) = 3$$

$$P_2 = 0.5(1)(-1) = -0.5$$

$$P_3 = -0.5(2)(1) = -1$$

so that

$$y(2.5) = -(3/8) [(9/3) - (18/0.5) - (31/1)] = 24.$$

4.20 Significant figures in polynomial representations. When an equation of the form

$$(4.20-1a) \quad y = \sum a_k f_k(x) \quad (k = 0, 1, 2, \dots)$$

is fitted to a set of experimental data, over the range $x_1 \leq x \leq x_2$, a decision is required on the number of digits that are significant in representing each a_k . From the relation

$$(4.20-2) \quad (\partial y / \partial a_k) = f_k(x)$$

one may deduce that a change in y of magnitude δy would be produced by a change in a_k of $\delta y / f_k$. If δy is chosen as $e_{\sigma 0}(y)$, a reasonable representation will result if a_k is stated so that the least significant decimal digit represents a quantity on the order of

$$(4.20-3a) \quad 0.01 e_{\sigma 0}(y) / |f_k|_{\max}$$

where $|f_k|_{\max}$ is the largest absolute value of f_k that occurs in the range $x_1 \leq x \leq x_2$.

The multiplier 0.01 represents a conservative allowance for the accumulation of errors in Eq. (4.20-1) and in subsequent computations with it. Very often, a trial computation will show that it is excessively conservative.

In particular, if $f_k = x^k$, so that Eq. (4.20-1a) becomes

$$(4.20-1b) \quad y = \sum a_k x^k, \quad (k = 0, 1, 2, \dots)$$

then the expression (4.20-3a) becomes

$$(4.20-3b) \quad 0.01 e_{\sigma 0}(y) / |x^k|_{\max}.$$

Example 4.20. The representation of the emf U of a type E thermocouple in the temperature range $-270 \leq t \leq 0$ °C, relative to a 0 °C cold junction, is given by an equation like Eq. (4.20-1) with $k = 1, 2, \dots, 13$ and with $e_{00}(U) = 0.1 \mu\text{V}$. For a_8 , (4.20-3b) yields

$$0.01 \times 0.1/270^8 = 3.5 \times 10^{-23}$$

so that the least significant decimal bit in a_8 should represent 1×10^{-23} . (The value of a_8 is approximately -9×10^{-13} , so that a_8 should have 11 digits.) [See Note N4.21, which refers to Refs. 4-3 and 4-4.]

4.21 Improvement of computational convenience. When the variable x may acquire values very much larger than unity, a representation like

$$(4.21-1) \quad y = \sum a_k x^k \quad (k = 0, 1, 2, \dots)$$

has two characteristics that may impede or inconvenience practical computation:

(a) As k becomes larger, the number of significant digits required to represent a_k may become very large; and

(b) If x has dimensions, each a_k has different dimensions, and only a_0 has the dimensions of y .

If $x_1 \leq x \leq x_2$, these deficiencies may be attenuated by transforming the independent variable to

$$(4.21-2) \quad X = (2x - x_1 - x_2)/(x_2 - x_1)$$

so that X is a nondimensional quantity and $-1 \leq X \leq 1$. Then Eq. (4.21-1) is replaced by

$$(4.21-3) \quad y = \sum A_k X^k \quad (k = 0, 1, 2, \dots).$$

Each coefficient has the same dimensions as y and, in accordance with Sec. 4.20, the least significant decimal digit of A_k need not be smaller than 0.01 times the expected uncertainty in y . [Note N4.21]

If either Eq. (4.21-1) or Eq. (4.21-3) is deduced from data by the method of least squares, exactly the same values of $y(x_i)$ and $e_{00}(y)$ will be obtained, unless the computer used is deficient. The transformation (4.21-2) serves to improve computational convenience, not accuracy. Accuracy is established by the data, not by how the data are represented. However, the more convenient procedure does reduce the number of significant digits required in any computing program that is used.

4.22 Polynomial approximations of intricate functions. Many engineering applications require that a relatively simple idealized formula be modified to allow for deviations from the idealized condition. The accurate computation of these modifications may require elaborate operations and the use of a computer; nevertheless, the final result may often be only a modest modification of the original idealized formula. In such case, the final result, once it has been accurately computed, may be rephrased as a simple low-degree polynomial in the variables affecting the deviations from the idealized formula. For example, if these variables are u, v, w, \dots and if the idealized formula is

$$(4.22-1) \quad s = s_0(x, y, z)$$

the corrected formula may take the form

$$(4.22-2a) \quad s = s_0 (1 + f)$$

where

$$(4.22-2b) \quad f = f(x, y, z, u, v, w)$$

and $|f| \ll 1$.

When there are two independent variables, the examples of Sec. 4.16 are applicable to the development of such correction formulas. It is desirable that the independent variables in Eq. (4.22-2b) be actually measured or otherwise known variables, or else be deduced from parametric equations containing such variables.

An important conclusion of Eq. (4.22-2a) is that the accuracy of knowledge of s is of the same order of magnitude as the accuracy of knowledge of s_0 . Thus, in applying Eq. (3.28-3) or (3.28-4), it is only necessary to deal with the elementary expression s_0 rather than with the more complex expression of s .

If the function f can be expressed as a linear combination of a few undetermined constants, these constants can be determined by a least-squares procedure, like one of those outlined in the preceding sections. An advantage of such least-squares procedures is that they provide the value of $e_{\sigma 1}$ as a measure of the agreement between the exact relation and its approximation.

Example 4.22a. The compressibility factor Z of normal hydrogen gas over the temperature range $300 \leq T \leq 600$ K and pressure range $0 \leq p \leq 100$ atm has been tabulated (Ref. 4-5). The results may be approximated within 0.1 percent of Z by the formula

$$Z = 1 + p^{\circ}(a_0 + a_1T + a_2p + a_3pT)$$

where $a_0 = 8.17 \times 10^{-4}$; $a_1 = -8.12 \times 10^{-7}$; $a_2 = 4.30 \times 10^{-7}$; $a_3 = -3.34 \times 10^{-10}$.

Example 4.22b. The compressibility factor Z of air over the temperature range $700 \leq T \leq 2000$ K and pressure range $0.5 \leq p \leq 100$ atm has been tabulated (Ref. 4-5). The results may be approximated to within 0.0002 in Z by the formula

$$Z = 1 + p^{\circ}(a_0 + a_1T + a_2T^2 + a_3p + a_4pT)$$

where $a_0 = 5.650 \times 10^{-4}$, $a_1 = -2.948 \times 10^{-7}$,

$$a_2 = 5.52 \times 10^{-11}, \quad a_3 = 2.35 \times 10^{-7}, \quad a_4 = -1.682 \times 10^{-10}.$$

Example 4.22c. The viscosity η of N_2 , as tabulated in Ref. 4-5, over the ranges $300 \leq T \leq 1500$ K, $1 \leq p \leq 100$ atm, is given with an rms error of 3 percent by

$$\eta/\eta_0 = x^{a+bx} \cdot y^c$$

where η_0 is the viscosity at $T_0 = 273.16$ K, $x = T/T_0$, $y = p$ in atm, and $a = 0.715$, $b = -0.0142$, $c = 0.0109$.

Example 4.22d. The mass flow rate \dot{m} of a known gas through a nozzle of diameter d and area A in a horizontal pipe of diameter D may be deduced from measurements of the upstream pressure p , upstream temperature T , and pressure drop Δp across the nozzle. When $\Delta p \ll p$, elementary application of Bernoulli's principle yields $\dot{m} = \dot{m}_0$ where

$$(4.22-3a) \quad \dot{m}_0^2 = (2p^{\circ}\Delta p/T)(1/R)(A^2)$$

and R is the specific gas constant. The first factor in parentheses on the right side of this equation represents the actual pressure and temperature measurements; the second factor represents gas properties; the third factor represents geometric features.

In practice, especially where Δp is an appreciable fraction of p , corrections are required for the compressibility and other properties of the gas and for geometric features of the installation. The complete relation is

$$(4.22-3b) \quad \dot{m} = \dot{m}_0 KY / \sqrt{Z}.$$

The quantity K is termed the *flow coefficient* and depends on the design of the nozzle, on the location of the points between which Δp is measured, and on the Reynolds number

$$(4.22-3c) \quad N_{Re} = 4\dot{m}_0 / (\pi D \eta)$$

of the flow, where η is gas viscosity. The quantity Y is a correction for the facts that the flow is isentropic but not isothermal, and that the gas is compressible; Y depends on the specific-heat ratio γ , on $\Delta p/p$, and on the diameter ratio $\beta = d/D$. The quantity Z is the *compressibility factor* of the gas (sometimes called the *supercompressibility factor*) and corrects for the fact that the gas is “real” rather than “ideal,” so that

$$(4.22-3d) \quad p = \rho ZRT.$$

Values of K and Y have been tabulated for a variety of standardized nozzle designs (Ref. 4-6). For any one nozzle design and any one gas, Eq. (4.22-3b) may usually be written as

$$(4.22-3e) \quad \dot{m} = \dot{m}_0 K_0 (1 - F)$$

where K_0 is a constant and $|F| \ll 1$. The percentage accuracy of knowledge of \dot{m} is of the same order of magnitude as the percentage accuracy of knowledge of \dot{m}_0 . Thus, a 1-percent uncertainty in measuring p , Δp , or T will result in about 0.5-percent uncertainty in \dot{m} ; a 1-percent uncertainty in each of the three independent measurements will result in about a 1-percent uncertainty in \dot{m} .

NOTES FOR CHAPTER 4

N4.3.1 Each of the following curves passes through the points $(0,1)$, $(1/\sqrt{2}, 1/\sqrt{2})$, $(1,0)$ with $\delta y < 0.001$:

1. $y = \cos(\pi x^2/2)$
2. $y = ax - b \sinh x + e^{-x}$; $a = 8.41$, $b = 7.47$
3. $x^2 + y^2 = 1$
4. $4(a-x)(a-y) = 1$; $a = (1 + \sqrt{2})/2$
5. $y = 1 + x - 2x^2$
6. $y = 1 - \alpha x^2 + (\alpha - 1)x^4$; $\alpha = 3 - 2\sqrt{2}$
7. $y = 1 + \alpha x + (\alpha - 1)x^3$; $\alpha = 3 - 2\sqrt{2}$

Furthermore, over the range $0 \leq x \leq 1$, the average disagreement between the ordinates of the first two curves is 0.004, and the maximum disagreement is less than 0.014.

N4.3.2 As an example, over the range $0 \leq x \leq 2$, the exponential $y = e^{-x}$ is approximated by the quadratic parabola

$$y(x) = 0.9813 - 0.8155x + 0.2000x^2$$

so that the difference between the ordinates of the two curves does not exceed 0.02. However, at $x = 2.5$, the difference is 0.11 and at $x = 3$, the difference is 0.28.

N4.8 The log scale of semilog (or log) paper is ordinarily marked with the values of y (or of y and of x). However, when the span of $\log y$ (or $\log x$) is a fraction of one decade, it may be more convenient to plot $\log y$ (or $\log x$) on a uniformly spaced scale of coordinates.

N4.10.1 Occasionally, the rms uncertainty $e_o(a_j)$ will also be stated. However, the criterion for determining the number of significant figures to be used in representing any a_j should be the criterion described in Sec. 4.20.

N4.10.2 A useful property of the least-squares method is that approximations made in the assignment of weights, for reasons of mathematical convenience, usually cause only small and acceptable increases in the value of e_{o1} or E_{o1} . However, in this text, no such approximations will be made unless they are explicitly identified.

N4.11 The two equations for e_{o1}^2 or E_{o1}^2 are alternate expressions of the same quantity. The advantage of the first expression is that the first summation on the right side can be computed at the time that the original data are entered into any computer program; the disadvantage is that the right side represents a very small difference between two relatively large quantities, so that each of these quantities must be computed with very high arithmetic accuracy. The advantage of the second expression is that it represents the direct summation of very small quantities, so that high arithmetic accuracy is not required; the disadvantage is that the original data must be reentered into a computer program after the simultaneous equations have been solved.

N4.12.1 Still another way of defining r is

$$(N4.12-1a) \quad r^2 = 1 - S^2(y)/\sigma^2(y) = 1 - S^2(x)/\sigma^2(x)$$

where

$$(N4.12-1b) \quad S^2(y) = \sum (a_0 + a_1x_i - y_i)^2 w_i = d_0 - a_0c_0 - a_1c_1$$

$$(N4.12-1c) \quad S^2(x) = \sum (A_0 + A_1y_i - x_i)^2 w_i$$

$$(N4.12-1d) \quad \sigma^2(y) = \sum (y_i - y_0)^2 w_i = d_0 - c_0^2/b_0$$

$$(N4.12-1e) \quad \sigma^2(x) = \sum (x_i - x_0)^2 w_i$$

$$(N4.12-1f) \quad y_0 = \sum (w_i y_i) / \sum w_i; \quad x_0 = \sum (w_i x_i) / \sum w_i$$

The b_k, c_k, d_k are given by (4.11-3), and all summations are for $i = 1, 2, \dots, n$.

N4.12.2 R.A. Fisher (Ref. 4-1) has pointed out that the distribution of errors in r , especially when r is close to unity, is not of Gaussian form and is skewed (i.e., it is not symmetrical about the most probable value). However, the transformation

$$(N4.12-2) \quad z = (1/2)\ln[(1+r)/(1-r)] = \tanh^{-1} r$$

leads to an approximately Gaussian distribution of z , so that $e_o(z)$, $e_a(z)$, and $e_p(z)$ are related by (3.18-2). The nonlinearity of Eq. (N4.12-2) is so pronounced that a change $\Delta z \equiv e_o(z)$ produces a change Δr in r that is radically different from the change $-\Delta r$ produced by a change $-\Delta z$. When $|r| > 0.8$ and the n pairs of data are equally weighted, this nonlinearity is adequately accounted for by Eq. (4.12-5).

The proportionalities between the uncertainties in z are

$$e_o(z): e_a(z): e_p(z) = 1: 0.80: 0.67$$

where, as deduced by Fisher,

$$e_o(z) \approx \beta.$$

N4.12.3 If

$$(N4.12-3) \quad y = f(x), \quad x = F(y)$$

are nonlinear functions of x or y , the correlation coefficient r is no longer useful. However, two other criteria of correlation may be used. They are the indices of correlation ρ_x and ρ_y defined by

$$(N4.12-4a) \quad \rho_y^2 = 1 - S^2(y)/\sigma^2(y)$$

$$(N4.12-4b) \quad \rho_x^2 = 1 - S^2(x)/\sigma^2(x)$$

where

$$(N4.12-4c) \quad S^2(y) = \sum [f(x_i) - y_i]^2 w_i$$

$$(N4.12-4d) \quad S^2(x) = \sum [F(y_i) - x_i]^2 w_i$$

$$(N4.12-4e) \quad \sigma^2(y) = \sum (y_i - y_0)^2 w_i$$

$$(N4.12-4f) \quad \sigma^2(x) = \sum (x_i - x_0)^2 w_i$$

$$(N4.12-4g) \quad y_0 = \sum (w_i y_i) / \sum w_i; \quad x_0 = \sum (w_i x_i) / \sum w_i$$

These equations are parallel in form to Eqs. (N4.12-1). The distinction between the two groups of equations is that ρ_x and ρ_y are usually different; they are equal to each other only when $f(x)$ has the linear form of Eq. (4.12-1).

The index of correlation may be used to determine whether a variable x is better correlated with y than is some other variable ξ . The comparison is possible only when the functional form f remains the same. Thus, if

$$y = f(x) \text{ yields } \rho_y = 0.9$$

and

$$y = f(\xi) \text{ yields } \rho_y = 0.95$$

then the correlation between ξ and y is better than the correlation between x and y .

However, for any set of n points (x_i, y_i, w_i) , the value of ρ_y may always be made to be closer to unity by increasing the number of undetermined constants in $f(x)$; if the number of these constants is equal to n , then ρ_x and ρ_y are identically equal to unity.

N4.14.1 The uncertainty of principal interest is $e_{o0}(y)$, or $E_{o0}(y)$, which represents, respectively, the rms uncertainty in y or the rms fractional uncertainty in y , which may be expected in future use of Eq. (4.14-1) in the x -interval (span) of the original data. These uncertainties are most readily expressed in terms of the quantity, $e_{o1}(y)$ or $E_{o1}(y)$, which represents the rms value of the residuals

$$(N4.14-1) \quad v_{yi} \equiv v(y_i) = f(x_i, a, b) - y_i$$

or

$$(N4.14-2) \quad V_{yi} \equiv V(y_i) = [f(x_i, a, b)/y_i] - 1,$$

the latter residual being used when, *a priori*, all y_i are deemed to have constant fractional uncertainty. Then

$$(N4.14-3a) \quad e_{\sigma 0}^2(y) = e_{\sigma 1}^2 / \sum w_i$$

where

$$(N4.14-3b) \quad (n-2)e_{\sigma 1}^2 = \sum v_{yi}^2; \quad e_{\sigma 1} \equiv e_{\sigma 1}(y);$$

or

$$(N4.14-4a) \quad E_{\sigma 0}^2(y) = E_{\sigma 1}/n$$

where

$$(N4.14-4b) \quad (n-2)E_{\sigma 1}^2 = \sum V_{yi}^2; \quad E_{\sigma 1} \equiv E_{\sigma 1}(y).$$

A convenient substitute for Eq. (N4.14-3b) or Eq. (N4.14-4b), in certain cases, is obtained by using the intermediate quantity $e_{\sigma 1}(s)$, given by

$$(N4.14-5) \quad (n-2)e_{\sigma 1}^2(s) = d_0 - Ac_0 - Bc_1$$

where

$$(N4.14-6) \quad d_0 = \sum (W_i s_i^2).$$

Examples of such cases are the following:

1. When the transformation is such that $s = y$, then

$$e_{\sigma 1} \equiv e_{\sigma 1}(y) = e_{\sigma 1}(s).$$

2. When the transformation is such that $s = \ln y$, and all y_i are deemed to have the same fractional error, then

$$E_{\sigma 1} \equiv E_{\sigma 1}(y) = e_{\sigma 1}(s).$$

3. When the transformation is such that $s = 1/y$, and all y_i are deemed to have the same fractional error, then

$$E_{\sigma 1} \equiv E_{\sigma 1}(y) = E_{\sigma 1}(s) = e_{\sigma 1}(s).$$

The relative merits of using Eq. (N4.14-5) in place of Eq. (N4.14-3b) or (N4.14-4b) are discussed in Note N4.11.

N4.14.2 The uncertainties of principal interest, $e_{\sigma 0}(y)$ and $E_{\sigma 0}(y)$, which represent the uncertainties to be expected in future use of Eq. (4.14-13) in the x -interval (span) of the original measurements, are describable in terms of the residuals

$$(N4.14-7) \quad v_{y_i} \equiv v(y_i) = f(x_i, a, b, c) - y_i$$

or

$$(N4.14-8) \quad V_{y_i} \equiv V(y_i) = [f(x_i, a, b, c)/y_i] - 1.$$

Then

$$(N4.14-9a) \quad e_{\sigma_0}^2(y) = e_{\sigma_1}^2 / \sum w_i; \quad e_{\sigma_1} \equiv e_{\sigma_1}(y)$$

where

$$(N4.14-9b) \quad (n-3)e_{\sigma_1}^2 = \sum (v_{y_i}^2 w_i) \quad ;$$

or

$$(N4.14-10a) \quad E_{\sigma_0}^2(y) = E_{\sigma_1}^2/n; \quad E_{\sigma_1} \equiv E_{\sigma_1}(y)$$

where

$$(N4.14-10b) \quad (n-3)E_{\sigma_1}^2 = \sum (V_{y_i}^2 w_i) \quad .$$

A convenient substitute for Eq. (N4.14-9b) or Eq. (N4.14-10b), in certain cases, is obtained by use of the intermediate quantity $e_{\sigma_1}(s)$, given by

$$(N4.14-11) \quad (n-3)e_{\sigma_1}^2(s) = d_0 - Ac_0 - Bc_1 - Cc_2$$

where

$$(N4.14-12) \quad d_0 = \sum (W_i s_i^2).$$

Three such cases are those described in Note N4.14.1:

1. If $s = y$, then $e_{\sigma_1} \equiv e_{\sigma_1}(y) = e_{\sigma_1}(s)$.
2. If $s = \ln y$, and $\delta y/y$ is presumed constant, then $E_{\sigma_1} \equiv E_{\sigma_1}(y) = e_{\sigma_1}(s)$
3. If $s = 1/y$, and $\delta y/y$ is presumed constant, then $E_{\sigma_1} \equiv E_{\sigma_1}(y) = E_{\sigma_1}(s) = e_{\sigma_1}(s)$.

The relative merits of using Eq. (N4.14-11) in place of Eq. (N4.14-9b) or (N4.14-10b) are discussed in Note N4.11.

N4.15.1 If each measurement is presumed to have the same uncertainty, then w_i should be set equal to unity, so that $\sum w_i = n$.

If each measurement is presumed to have the same fractional uncertainty, then w_i should be replaced by $1/y_i^2$, so that $\sum w_i y_i^2 = n$.

N4.15.2 Each of the coefficients a_k has an rms uncertainty $e_\sigma(a_k)$. A general procedure for finding these is indicated in Note N4.16. Only a change in the formal symbolism is required.

N4.16 Each of the coefficients a_k has an rms uncertainty $e_\sigma(a_k)$. The procedure for finding these uncertainties is as follows. Let D represent the determinant of the coefficients on the left side of the set of equations (4.16-3)

$$D = \begin{vmatrix} b_{11} & \dots & b_{1m} \\ \dots & \dots & \dots \\ b_{m1} & \dots & b_{mm} \end{vmatrix}$$

and let D_{kk} represent the cofactor of the diagonal term b_{kk} . Then

$$e_\sigma^2(a_k) = (D_{kk}/D)e_{\sigma 1}^2.$$

This equation is exact if the weights do not depend on the y 's, as in Case A, Sec. 4.11; it is a close approximation if the weights do depend on the y 's, as in Case B, Sec. 4.11.

For example, in Example 4.11d, Case C,

$$e_\sigma^2(a_0) = e_{\sigma 1}^2 \bullet (b_2 b_4 - b_3^2) / D$$

$$e_\sigma^2(a_1) = e_{\sigma 1}^2 \bullet (b_0 b_4 - b_2^2) / D$$

$$e_\sigma^2(a_2) = e_{\sigma 1}^2 \bullet (b_0 b_2 - b_1^2) / D$$

where

$$D = \begin{vmatrix} b_0 & b_1 & b_2 \\ b_1 & b_2 & b_3 \\ b_2 & b_3 & b_4 \end{vmatrix}$$

N4.17 Either Y or τ may be computed without knowledge of the other.

If Y is of principal interest, but τ is also of interest, then Y may be computed first by Eq. (4.17-5), and then Eq. (4.17-7b) may be replaced by

$$(N4.17-1) \quad p = (Yb_1 - c_1)/b_2$$

If τ is of principal interest, but Y is also of interest, then τ may be computed first by Eq. (4.17-7), and then Eq. (4.17-5) may be replaced by

$$(N4.17-2) \quad Y = (c_0 + pb_1)/n$$

Use of Eqs. (N4.17-1) and (N4.17-2) involves fewer computational steps than use of the equations they replace. This fact may be significant when a real-time (on-line) computer program is to be used.

N4.21. A further moderate reduction in the number of significant figures is possible by replacing Eq. (4.21-3) with

$$(N4.21-1) \quad y = \sum \alpha_k T_k$$

where $T_k(X)$ is the Chebyshev polynomial defined by

$$T_{k+2} = 2XT_{k+1} - T_k \quad (k = 0, 1, 2, \dots, m)$$

$$(N4.21-2) \quad T_1 = X$$

$$T_0 = 1$$

and X is given by Eq. (4.21-2). The Chebyshev polynomial has the property that

$$-1 < T_k < 1 \quad \text{if} \quad -1 < X < 1$$

The advantage of Eq. (N4.21-1) over Eq. (4.21-3) becomes more significant as m becomes larger.

As an example, the emf y , in microvolts, generated by a type E thermocouple, when the reference junction is at 0 °C and the other junction is at x °C, in the range $-270 \leq x \leq 0$ °C, is given to within 0.01 microvolt by Eqs. (4.21-1), (4.21-3), and (N4.21-1) when the coefficients have the following values:

k	a_k	A_k	α_k
0	0	-6713.695	-5830.645
1	58.695 857 799	5278.258	5015.620
2	0.051 667 517 705	1685.958	897.483
3	-4.465 268 3347 $\times 10^{-4}$	-296.232	-94.451
4	-1.734 627 0905 $\times 10^{-5}$	131.536	15.190
5	-4.871 936 8427 $\times 10^{-7}$	-71.680	-3.898
6	-8.889 655 0447 $\times 10^{-9}$	-279.488	0.563
7	-1.093 076 7375 $\times 10^{-10}$	154.880	0.360
8	-9.178 453 5039 $\times 10^{-13}$	710.272	-0.265
9	-5.257 515 8521 $\times 10^{-15}$	-505.088	0.064
10	-2.016 960 1996 $\times 10^{-17}$	-689.664	0.045
11	-4.950 213 8782 $\times 10^{-20}$	574.464	-0.128
12	-7.017 798 0633 $\times 10^{-23}$	237.568	0.116
13	-4.367 180 8488 $\times 10^{-26}$	-217.088	-0.053

The original data and the coefficients a_k are from Ref. 4-3. The coefficients α_k are from Ref. 4-4, which pointed out the usefulness of the Chebyshev polynomials.

CHAPTER 5. SOME BASIC ELEMENTS OF STATIC PERFORMANCE

5.0 Terms characterizing instrument performance. Some terms that may be applicable to characterization of an instrument's performance are listed here.

A. Characteristics expressed in units of the physical variable being measured.

- (1) *Range*: a statement of the limits y_{\min} and y_{\max} . (Sec. 3.4)
- (2) *Span*: the value of $|y_{\max} - y_{\min}|$. (Sec. 3.4)
- (3) *Readability*: the smallest change in indication that can be detected.
- (4) *Repeatability*: the extent to which the same indication will be obtained when the value of the physical variable is altered and then restored to its original value. The nature of the alteration must be stated. Repeatability may be different in the following different circumstances:
 - (a) the reading is made immediately after restoration.
 - (b) the reading is made several minutes after restoration.
 - (c) the reading is made several days after restoration.
- (5) *Sensibility*: the smallest change that can be detected reliably. This value is usually larger than the readability. It is often synonymous with repeatability. It is a convenient descriptive term, even though its definition is not precise.
- (6) *Inaccuracy*: the difference between the indication and the true value of the quantity it is intended to measure, as treated in Chapter 3. It usually represents the random error of measurement and is stated as the probable, average, or rms error (standard deviation), or as a percentile (Sec. 3.20), or as a limit of error (Sec. 3.7).

B. Other characteristics.

- (7) *Sensitivity*: the value dy/dx , where x represents the physical variable being measured, and y represents the indication. This indication may be in any one of various units: scale divisions, counter readings, voltage, current, pressure, or the same as the units of x .¹
- (8) *Nonlinearity*: in instruments whose indication y is intended to be proportional to some function u of the physical variable x being measured, the extent to which that proportionality is actually achieved. It is usually described either as
 - (a) the variation in dy/du over the range of the instrument, or
 - (b) the maximum difference between the indication and the ordinate of a straight line approximating $y(u)$, both values taken at the same abscissa. The various ways in which the straight line may be drawn have been treated in Sec. 4.9. The maximum error may occur at a limit of the range or at some intermediate point. The nonlinearity is usually expressed as a percent of range or at some intermediate point. The nonlinearity is usually expressed as a percent of span, or, if the lower limit of the range is zero, as a percent of full scale.

¹Industrial-process instruments in the U.S.A. are often designed to provide standardized indications like 1-5 V, 4-20 mA, or 3-15 psig, in order to facilitate operation of a variety of control elements of different manufacturers. The lower limit of these indications corresponds to the value $x = 0$.

5.1 Pointer and scale instruments. Such instruments are useful when

- (a) the variable being observed fluctuates slightly and randomly with time, and knowledge of the existence, periodicity, and order of magnitude of the fluctuation is valuable;
- (b) a particular value of the indication (the "set point") is desired and it is of value to know not only whether or not the set point has been reached, but also the degree of deviation from the set point and, if the deviation is unsteady, how it varies with time. Instead of a set point, there may be a "working band"—a band of contiguous values.
- (c) rapid visual perception of the approximate indication is necessary
- (d) there are advantages in cost, reliability, or size over instruments using counters, lights, or flags.

Example 5.1a. When a null galvanometer is used to indicate the balance point of a Wheatstone bridge, the approach to the balance point may be made smoothly and any drift in indication due to self-heating may be detected.

Example 5.1b. When a number of 300-degree or 360-degree circular scale instruments is used on an instrument panel, an appropriate group of instruments may be arranged so that normal (bogey or nominal) indication is represented by the 3-o'clock position of each pointer. Any deviation from the bogey conditions is then quickly apparent.

Example 5.1c. In the flight test of an airplane during an acrobatic maneuver, the pilot may be asked to attain and hold a desired acceleration for a brief period of time. An indicating accelerometer can provide the required indication of the approach, maintenance, and recovery from this condition. The accelerometer becomes part of a feedback control system using a human operator. [Note N5.1]

5.2 Scales. When accuracy is important and quick readability is not, the following sources of error may be present in mass-produced scales. Their relative prominence affects the nature and extent of the instrument calibration that is required.

- (1) Play between pivot and socket, in instruments using this more rugged construction
- (2) For elastically supported pointers, the relevant spring errors listed in Sec. 5.6
- (3) The fact that the pivot is not at the center of a circular scale, dial, or drum, so that a systematic error results
- (4) Improper location of major scale intervals²
- (5) Improper subdivision of major intervals, involving:
 - (a) systematic errors caused by an incorrect interpolation process
 - (b) random errors caused by the marking machinery
- (6) Shrinkage and distortion caused by a duplicating process
- (7) Parallax in reading:
 - (a) For scales without mirrors, the error due to parallax that may reasonably be expected is a distance on the scale equal to one-fifth of the distance between pointer and scale.
 - (b) For scales with mirrors and knife-edge pointers, the parallax error is negligible in comparison with the error listed under "inadequate readability."
- (8) Inadequate readability. A working rule is that, neglecting parallax, the readability is not better than the largest of the following three quantities:
 - (i) a distance of 0.1 mm on the scale
 - (ii) one-fifth of the thickness of a knife-edge pointer
 - (iii) one-tenth of the smallest scale interval between adjacent markings

5.3 Force measurements. Physical quantities in mechanics and electricity whose dimensional representation in the force-length-time system explicitly includes the dimension of force are often measured, directly or

²Usually, a precision instrument is marked to be correct at major scale intervals, during the manufacturer's calibration or adjustment. Subdivision between these intervals is then performed by a machine in accordance with a predetermined interpolation rule.

indirectly, through a force measurement. This measurement, in turn, is commonly made by one of the following methods:

- A. Comparison with the force exerted by gravity on a known mass (lever-type balance, or pendulum)
- B. Measurement of the current through a coil or solenoid in the field of a permanent magnet (electrobalance)
- C. Measurement of the deflection of a spring.

5.4 Errors in dynamometers using gravity. Measurements using method 5.3A are subject to errors caused by

- (a) uncertainty in the knowledge of the value of the local acceleration of gravity [Note N5.4]
- (b) incorrect length of lever arms, because of imperfections in manufacture
- (c) friction in bearings or knife edges
- (d) play in bearings or knife-edge seats
- (e) bending of levers under load
- (f) systematic change in lever-arm geometry with load
- (g) change in lever-arm lengths caused by temperature or by temperature gradients
- (h) uncertainty or systematic change in pivot-point location when elastic knife edges (Emory knife edges) are used
- (i) resisting torque or force when elastic knife edges are used
- (j) uncertainty in location of the effective line of action when flexible tapes are used in contact with pulleys or cams, because of flexural resistance of the tape or of presence of dirt between the tape and the surface of the cam
- (k) uncertainty of effect of air buoyancy on the components
- (l) electrostatic forces on components

5.5 Errors in electrobalances. Electrobalances (method 5.3B) intrinsically depend on the relation between force and coil current. This current is usually determined by measuring the voltage drop across a stable precision resistor in series with the coil. The measurements are subject to errors caused by

- (a) the systematic effect of temperature on the remanence of the permanent magnet³
- (b) change in coupling between the coil or solenoid and the magnet, due to self-heating or change in the direction of gravitational acceleration (if the electrobalance is not of the rotary type)
- (c) reduction of magnet strength (demagnetization) caused by mechanical impact, excessive vibration, or an external magnetic field⁴
- (d) any of the sources of error listed in Sec. 5.4 for method 5.3A, that may be applicable

5.6 Errors in springs. Instruments using method 5.3C are usually designed to have a substantially linear relation between the scale indication and the quantity being measured. Terms specifying the performance of instruments in this group are

- (a) *Nonlinearity.* See item (8) in Sec. 5.0.
- (b) *Hysteresis.* In its most general sense, the greatest difference between the ordinates, at the same input, of the two curves of indication versus input, one curve generated by steadily increasing the input from its lower limit to its upper limit, the other curve generated in the reverse direction and also over the full span. This general definition includes three effects:
 - (i) backlash
 - (ii) dry friction⁵
 - (iii) true elastic hysteresis of the spring material

³Properly prepared Alnico may have a temperature coefficient of remanence smaller than 0.0003/°C.

⁴Properly prepared Alnico is highly resistant to such influences.

⁵Although the coefficient of friction may be low when two surfaces are in relative motion, the coefficient may be many times larger at the instant when the motion is initiated from a position of rest. This phenomenon is evident even in lubricated ball, roller, and journal bearings.

To most users, only the combined result of these three effects is of interest. [Notes N5.6.1 and N5.6.2]

Hysteresis, like nonlinearity, is usually described as a percent of span.

- (c) *Zero drift*. The difference between the indication immediately after removal of load (generally full-scale load) and the “final” indication some time later. It is usually expressed as a percent of span. [Notes N5.6.2 and N5.6.3]
- (d) *Full-scale drift (creep)*. The difference between the indication immediately after application of full-scale input and the “final” indication some time later. It is usually expressed as a percent of span. [Notes N5.6.2 and N5.6.3]
- (e) A common, convenient specification is that combined nonlinearity, hysteresis, and drift of an instrument shall not exceed a stated percent of the span (or of full scale). If the span is much larger than full scale, it is necessary to specify explicitly whether hysteresis should be based upon a change equal to full scale or upon a change equal to the span.

Since nonlinearity is a systematic error, it may be corrected for with slight loss of convenience but with gain in accuracy. However, hysteresis and drift must be considered as random errors, since their sign and magnitude depend on the prior history of change in the input.

- (f) *Temperature coefficient of sensitivity*. With sensitivity defined as in Sec. 5.0.B.7, this property may be described in either of the following ways:
 - (i) the change in sensitivity, per unit change in temperature, at the nominal (bogey) operating temperature, or
 - (ii) the change in sensitivity over a stated range of operating temperatures; this is also expressible as a mean coefficient over the stated range.

For a linear spring, the temperature coefficient of sensitivity is usually the temperature coefficient of the modulus of elasticity of the spring material plus the temperature coefficient of linear expansion of the material. [Note N5.6.4]

- (g) *Temperature effect on zero*. The principal component of this effect is usually a systematic variation with temperature, but there may also be a random component that depends on the prior temperature history of the spring. The systematic effect may be described in one of the following ways:
 - (i) the derivative of zero shift with respect to temperature at the nominal (bogey) operating temperature
 - (ii) the average change in zero, per unit temperature increment, over a stated temperature range.

An overall specification that includes both systematic and random effects is the maximum shift of zero when the temperature is varied over a stated range in any arbitrary manner.

- (h) *Pressure effects*. Ambient pressure may affect both sensitivity and zero. Systematic effects can be described as a percentage change in sensitivity and an absolute change in zero, per unit change in pressure; such effects may be corrected for. (Differential-pressure gauges often show such effects.) Random effects are best described as the maximum change in sensitivity or zero over a stated range of ambient pressure.⁶
- (i) Some items listed in Sec. 5.4 apply also to dynamometers using springs.⁷

5.7 External physical variables affecting steady-state indication. The environment in which the instrument is used may affect its indication or operation. Among such environmental factors are

- (1) Temperature and temperature gradients
- (2) Ambient or barometric pressure
- (3) Steady acceleration (e.g., position or attitude relative to the Earth)
- (4) Vibration or acoustic noise

⁶Mechanical oscillators, like tuning forks, if not hermetically encapsulated, are subject to effects of ambient air pressure, but these effects are due principally to the change in mass of entrained gas rather than to the change in spring stiffness or in dimensions.

⁷The terms *reproducibility* or *repeatability*, when applied to springs, usually include random effects, but exclude systematic ones which may be corrected for.

- (5) Heat exchange with surroundings (radiative, conductive, or convective)
- (6) Correlated electric or magnetic fields (hum) or currents, such as those due to ac power lines
- (7) Uncorrelated electric or magnetic fields (noise), such as those created by adjacent instruments; *radiofrequency interference (RFI)* is one appellation for such fields in the radio spectrum
- (8) Optical, X-, γ -, neutron, or cosmic radiation
- (9) Ambient-gas composition, including humidity
- (10) In electrically powered instruments, deviations in the power supply from the nominal (bogey) values of voltage, frequency, or wave shape
- (11) In pneumatically powered instruments, deviations from the nominal value of gas pressure, or the presence of particulates (dust) or water in the gas supply.

An instrument user may either

- (a) specify the limits of any one of these factors, between which satisfactory operation will be assured, or
- (b) specify limits on the magnitude of any systematic effects of these factors, or
- (c) require knowledge of the quantitative magnitude of an effect, so that a systematic correction may be applied by the user, or
- (d) take corrective measures to reduce an effect to an acceptably small magnitude.⁸

Example 5.7a. A specification for an electrically powered instrument may require that the change in indication shall not exceed a specified amount when there is any combination of line voltage between 105 and 130 volts, line frequency between 59 and 61 hertz, and harmonic distortion between 0 and 5 percent.

Example 5.7b. A specification for a differential-pressure gauge with a range of 1 unit of pressure may require that the effect of ambient temperature shall not exceed 0.0005 unit/°C in the range 10 to 30 °C, that the effect of ambient pressure shall not exceed 0.01 unit/bar in the range 0.5 to 1.5 bar, and that a 30-degree tilt of the gauge in any direction from its normal operating attitude shall not change the indication by more than 0.001 unit.

Example 5.7c. A specification may require a statement of the mean sensitivity coefficients for each of the effects listed in Example 5.7b.

Example 5.7d. For a pneumatically powered instrument, the user may provide a pressure regulator, a filter, and a dryer for the gas supply.

Example 5.7e. For an instrument intended to measure low-level emf's, the manufacturer and the user may take steps

- (i) to reduce the presence and the effects of thermal emf's [Notes N5.7.1 and N5.7.2]
- (ii) to reduce the electrically and magnetically induced effects of alternating-current power lines, particularly if these provide power for the instrument [Note N5.7.3]

5.8. Long-term effects of external physical variables. Sometimes, the effect of an external physical variable may become significant only after a long time of exposure. Such long-term effects cannot usually be included in an instrument specification, but their presence and magnitude influence instrument selection because they affect the frequency with which recalibration and routine maintenance will be required.

Example 5.8a. The indication of a thermocouple in an installation where a section ΔL has been in a high-temperature gradient $\Delta\theta/\Delta L$ for a long period of time may change significantly after the gradient has been removed or after the section ΔL has been replaced by a new section because the immersion depth has been changed.

⁸Approach (c) usually yields reductions in cost and in acquisition time over approaches (a) and (b), but it also involves losses in convenience and in ease of data analysis and use.

Example 5.8b. The calibration of a thermocouple which has Rh or Re as a component of one of its wires may have changed significantly after months or years of exposure to a high thermal neutron flux.

Example 5.8c. Oxidation of the surface of a radiation shield or of a pyrometer wire may alter the radiation correction of a pyrometric or radiometric device.

Example 5.8d. Accretion of dirt on the window of an optical pyrometer, nephelometer, or radiometer may alter the instrument's indication.

Example 5.8e. Accretion of dirt on the surface of a nozzle or of a pipe may alter the calibration factor of a head-type fluid flowmeter or of a boundary-layer-type calorimetric fluid flowmeter.

5.9. Effects of the spatial profile. If there is a spatial variation of a physical quantity u , the spatial integral of u (the integral of u over a linear distance, area, or volume) may be of interest in itself or may be used to provide the average \bar{u} of u . Such an integral or average may be obtainable by one or a combination of the following methods.

- (1) A single local measurement of u at a location at which u has the same value as \bar{u} . This method is useful when the profile is known.
- (2) Use of an instrument that directly provides an indication that represents the correct integral of u .
- (3) Measurement of the profile by a continuous traverse with an instrument that provides the local value of u . This method usually is practical only when the traverse is along just one of the orthogonal dimensions (e.g., straight line or arc) of the relevant coordinate system and when the local values of u change negligibly during the time interval consumed by the traverse.
- (4) Simultaneous discrete local measurements of u at locations chosen so that the summation of these measurements, with appropriate weighting, yields the desired integral.

Each of these methods will be treated in the sections that follow. The spatial distribution of u may be represented

- (a) in rectangular Cartesian coordinates, by $u(x)$, $u(x,y)$, or $u(x,y,z)$;
- (b) in polar or cylindrical coordinates, by $u(r,\theta)$, or $u(r,\theta,z)$.

5.10. Spatial average from a single local measurement. When it is known that a profile $u(x)$ could be represented by a simple analytical formula involving only one constant or parameter that changes with the average velocity \bar{u} , there may be one location x_0 at which $u(x_0) = \bar{u}$ for all values of \bar{u} that may be experienced. It may be necessary to specify certain physical conditions that will ensure this favorable situation.

If the profile is expected to vary between two limiting forms, it may be possible to find some intermediate value of x_0 which will yield equal absolute values of the positive and negative errors that occur when the respective limiting profiles exist.

Example 5.10. To determine the volumetric velocity (bulk velocity) of fluid flowing unidirectionally in a circular pipe of radius R , if the fluid fills the pipe, and if the flow is axially symmetric, one may assume that the average linear velocity \bar{u} is equal to the local velocity at some radius r_0 whose value depends on the nature of the flow.

- (i) If the flow is fully developed and laminar, $u(r_0)$ will be equal to \bar{u} (bulk velocity/area) when

$$r_0/R = 1/\sqrt{2} \quad \text{[Note N5.10.1]}$$

- (ii) If the flow is fully developed and turbulent, $u(r_0)$ will be equal to \bar{u} when

$$r_0/R = 0.758 \quad \text{[Note N5.10.2]}$$

- (iii) Consequently, if the flow is expected to range from fully developed laminar to fully developed turbulent, a measurement of u at

$$r_0/R = 0.715$$

will yield a fractional error in mean-velocity measurement, such that

$$|[1 - u(r_0)/\bar{u}]| \leq 2.3 \% .$$

5.11 Spatial integration by physical means. An appropriate geometric configuration may be used to produce a measurable effect representing the desired integral.

Example 5.11a. The heat of combustion of a chemical mixture of known mass may be measured by enclosing the mixture in a thermally isolated calorimetric “bomb” of high thermal conductivity, igniting the mixture by an electric spark, and measuring the resultant temperature rise of the enclosure, whose thermal capacity is known. Systematic corrections must be made for the spark energy and for any imperfections in the thermal isolation.

Example 5.11b. The radiant flux from a surface of large area may be measured by optically imaging that surface upon an energy receiver at the focus of an appropriate optical system. The receiver may be a bolometer or a photocell so that the measurement actually made is a resistance change or an emission current. Rather than measuring and correcting for the transmission factor of the optics and the characteristics of the energy receiver, calibration is usually effected by viewing a black-body surface of the same area, held at a known temperature.

Example 5.11c. The convective heat flux delivered by an impinging gas stream to a portion of a surface may be measured by replacing that portion with a sheet of material of known thermal conductivity and measuring the temperature gradient in the material in a direction normal to the surface.

Example 5.11d. The mass flow rate of a fluid in a pipe is measured by use of a venturi in which pressure differential Δp and upstream pressure p and temperature T are measured. The basic product $K_0(p \Delta p / T)^{1/2}$, where K_0 is a calibration factor for certain nominal (bogey) conditions, must be modified by systematic corrections which were indicated in Example 4.22d.

Example 5.11e. The mass flow rate \dot{m} of a fluid in a pipe is measured by a Thomas-type of calorimetric flowmeter. A grid of heating wires uniformly distributed across the cross section of the pipe delivers heat to the fluid at the rate W ; two similar grids upstream and downstream of the heating grid act as resistance thermometer elements so that the temperature rise ΔT of the fluid may be determined. The assembly is thermally isolated so that the process is adiabatic. If c is the specific heat capacity of the fluid,

$$\dot{m}c \Delta T = W$$

A systematic correction is required for the variation of c with fluid temperature. The grids may not provide uniform heating or correct averaging of temperature over the cross section of the pipe, but a systematic correction may be correlated with N_{Re} if an empirical calibration is performed.

Example 5.11f. A liquid flows in a circular pipe of radius R in which there is fully developed axially symmetric flow. The average axially directed speed \bar{u} is to be deduced from an ultrasonic flowmeter in which a sound pulse is sent between two transducers located at the cylindrical coordinates $(-R, 0, z_1)$ and $(R, \pi, z_1 + L)$. If t_1 is the transit time when the sound travels in the direction of

the fluid flow, and t_2 is the transit time when the sound travels in the opposite direction, the velocity \bar{u} is to be deduced from the difference

$$\Delta t = t_2 - t_1 \quad .$$

If fluid density is constant everywhere and fluid velocity is very small compared to the speed of sound c_A in the fluid, then the formula giving the relation between \bar{u} and Δt is of the form

$$\bar{u} = Kc_A^2 \Delta t / (2L)$$

where K depends on the velocity profile. For a completely flat profile (fluid velocity the same at every location), $K = 1$; for a fully developed laminar profile, $K = 0.75$; for a fully developed turbulent profile, $K = 0.844$. Thus, if the velocity profile is known to be laminar ($N_{Re} < 1000$) or turbulent ($N_{Re} > 2000$), systematic corrections can be applied. However, if the Reynolds number lies in the transition region or if the velocity profile is not known for any reason, the assumption $K = 0.797$ would yield a fractional error not greater than 6 percent. The uncertainty may be larger if the profile is not fully developed or is distorted because of the piping configuration.

5.12 Profile determination by continuous traverse. An empirical determination of the profile by continuous traverse is usually desirable when a new instrument, measurement technique, or physical situation is first introduced. Once such a determination is made, subsequent repetitive measurements may be made by the simpler methods treated in Secs. 5.10 and 5.11, because the required calibration factor or coordinate usually called for by those methods has been established with greater certainty than is possible by purely theoretical analysis.

An appropriate procedure is initially to perform continuous traverses over the range of independent parameters that are, *a priori*, believed to influence the profile, thereby to establish the appropriate correction factors for the range of parameters, and then to replace the traversing method with the simpler single-measurement method. If the independent parameters are nondimensional quantities, the amount of work is usually minimized, and the utility of the results is augmented.

Because of its mechanical complexity, a continuous traverse is usually convenient only in the direction of that coordinate in which the measured variable's profile is least predictable. For example, in measuring the profile of fluid flow in a straight length of pipe, axial symmetry is likely and a radial traverse is more important than a circular one.

In the time that is taken to complete the traverse, it is necessary that the variable u being measured change less than the permissible uncertainty Δu in that measurement. On the other hand, the traverse may not be made so speedily that the instrument used in the traverse cannot follow the changes in u with distance x . If the traversing-instrument system has time constant τ (see Chs. 6 and 7) and the gradient of u is du/dx , then it is necessary that

$$(5.12-1) \quad dx/dt < (\Delta u/\tau)/(du/dx) \quad .$$

If (du/dx) varies appreciably over the distance of traverse, it is necessary either that

- (i) $(du/dx)_{\max}$ replace (du/dx) in Eq. (5.12-1), if dx/dt is constant, or
- (ii) dx/dt be varied so as to satisfy Eq. (5.12-1) over the entire range of traverse.

5.13 Integration or averaging by discrete local measurements. When the spatial variation of a physical variable u can be described by an analytic formula that contains n independent constants, the average value \bar{u} may be deduced from local measurements of u at n locations which depend on the form of the formula. The case $n = 1$ was treated in Sec. 5.10. When the spatial variation is not predictable, the approximate value of \bar{u} may be estimated from measurements at n discrete locations. If only one space dimension X is involved, so that $u = u(X)$, this estimation is equivalent to fitting a polynomial of degree n to $u(X)$. The extent to which any weighted sum of n measurements agrees with the integral of $u(X)$, over any stated

X -interval, depends on the choice of locations at which measurements of $u(X)$ are made and on the weights assigned to each measurement.

For any one coordinate X and any one interval

$$X_a \leq X \leq X_b,$$

four methods of selecting the measurement locations and of summing the measurements will be described:

- A. Equally weighted measurements at n equally spaced locations. (Centroid-of-equal-areas method). For any given n , this method is the simplest and least accurate of the four methods.
- B. Appropriately weighted measurements at n equally spaced locations. (Newton-Cotes method). This is the only one of the four methods which requires measurements at the extreme ends of the interval.
- C. Equally weighted measurements at n appropriately spaced locations. (Chebyshev method)
- D. Appropriately weighted measurements at n appropriately spaced locations. (Gauss method). For any given n , this method is the most accurate of the four methods.

Table 5-1⁹ lists locations and weights for various values of n and for the following conditions:

- (a) In rectangular coordinates, the range

$$X_a \leq X \leq X_b$$

has been normalized to

$$0 \leq x \leq 1$$

by the transformation

$$x = (X - X_a) / (X_b - X_a)$$

- (b) In polar coordinates, the range

$$0 \leq R \leq R_m$$

has been normalized to

$$0 \leq r \leq 1$$

by the transformation

$$r = R/R_m$$

[Note N5.13.1]

The number n must not be so large that the devices used to make the measurements significantly affect the distribution $u(X)$ or its spatial integral or its spatial average, unless these devices are a permanent part of the mechanical installation and exert a permanent influence on the profile of u . [Note N5.13.2].

5.14 Average value of a function of several measured variables. If a quantity z is computed from measurements of several variables $u_1, u_2, \dots, u_j, \dots$ that vary with some parametric variable x that represents spatial location or time, so that

⁹For ease of later reference, Table 5-1 appears at the end of this text on page 220. An earlier version of this table was prepared by the author for the ASME Power Test Code on Velocity Measurement, and appears also in Ref. 4-6.

$$(5.14-1) \quad z = z(u_1, u_2, \dots, u_j, \dots)$$

$$(5.14-2) \quad u_j \equiv u_j(x),$$

the technique for computing the space-averaged or time-averaged value z_0 of z over the interval $x_a \leq x \leq x_b$ depends on whether the space variation or time variation of the measured u_j is random or systematic. Here, z_0 may be considered as being given by

$$(5.14-3) \quad (x_b - x_a) \bullet z_0 = \int_{x_a}^{x_b} z \bullet dx \quad .$$

The variable x may represent a space variable like distance, area, or volume or a time variable like time or phase angle.

A. Random variation. If all of the measured variables vary randomly with respect to x , so that there is no correlation between the local or simultaneous values of any pair of the u_j , then, over the interval $x_a \leq x \leq x_b$, each u_j may be averaged to yield u_{j0} given by

$$(5.14-4) \quad (x_b - x_a) \bullet u_{j0} = \int_{x_a}^{x_b} u_j(x) \bullet dx \quad ,$$

and the average value of z computed as

$$(5.14-5) \quad z_0 = z(u_{10}, u_{20}, \dots, u_{j0}, \dots) \quad .$$

This procedure usually is the simplest, because it involves a minimum of instrumentation, computation, and data processing. Its applicability becomes more likely as the interval $(x_b - x_a)$ becomes smaller.

B. Systematic variation. If the u_j vary systematically with respect to space or time, so that there exist correlations among the local or instantaneous values of the u_j , it is necessary first to compute the local or instantaneous value of z , as given by Eq. (5.14-1), and then to compute z_0 by Eq. (5.14-3). This procedure is usually complex because it requires a sufficiently large number of computations of z by Eq. (5.14-1) to permit adequately accurate computation of the integral in Eq. (5.14-3).

C. Combined random and systematic variations. An intermediate level of complexity in instrumentation is achieved if some of the variables, say u_1 to u_k , are uncorrelated with each other or the other u_j , and only the variables u_{k+1}, \dots are correlated with each other. Then Eq. (5.14-4) is used to find $u_{10}, u_{20}, \dots, u_{k0}$ and these values are inserted into Eq. (5.14-1) to give

$$(5.14-6) \quad z = z(u_{10}, u_{20}, \dots, u_{k0}, u_{k+1}, u_{k+2}, \dots) \quad .$$

This computation may be easier than one in which all u_j vary with x , but the integration (5.14-3) must still be performed.

5.15 Error due to incorrect averaging. Since the averaging operation represented by Eq. (5.14-5) generally is much less complex than the operation represented by Eq. (5.14-3), the pragmatic question arises of when the simpler operation is an adequate substitute for the more complex one. If the measured quantities vary systematically with x , the error Δz_0 that is made by treating them as random is given by

$$(5.15-1) \quad \Delta z_0 = z(u_{10}, u_{20}, \dots, u_{j0}, \dots, u_{n0}) - \frac{1}{x_b - x_a} \int_{x_a}^{x_b} z(u_1, u_2, \dots, u_j, \dots, u_n) dx$$

and the fractional error, if small, is approximately $\Delta z_0 / z(u_{10}, u_{20}, \dots)$.

The magnitude of the error may be estimated by assuming a simple analytic or graphic approximation of the functions $u_j(x)$, on the basis of some preliminary experiments, previous experience, or other physical intuition, and then performing, analytically or graphically, the computation represented by Eq. (5.15-1).

Special case A. In the particular case where the quantity z is the product of two variables, so that

$$(5.15-2) \quad z = u_1 u_2,$$

the error Δz_0 is given more readily by the formula

$$(5.15-3) \quad (x_b - x_a) \Delta z_0 = - \int_{x_a}^{x_b} (u_1 - u_{10})(u_2 - u_{20}) dx .$$

In any case where the variables u_1 and u_2 vary randomly and there is no correlation between them, the integral in Eq. (5.15-3) approaches zero as the interval $(x_b - x_a)$ becomes larger.

Special case B. In the particular case where the quantity z is the product of three variables, so that

$$(5.15-4) \quad z = u_1 u_2 u_3$$

the error Δz_0 defined by Eq. (5.15-1) is given more readily by the formula

$$(5.15-5a) \quad (x_b - x_a) \Delta z_0 = - \int \Delta u_1 \Delta u_2 \Delta u_3 dx - u_{10} \int \Delta u_2 \Delta u_3 dx \\ - u_{20} \int \Delta u_1 \Delta u_3 dx - u_{30} \int \Delta u_1 \Delta u_2 dx$$

where

$$(5.15-5b) \quad \Delta u_k \equiv u_k - u_{k0} \quad k = 1, 2, 3$$

and all integrals are from x_a to x_b .

Special case C. In the particular case where the quantity z is given by

$$(5.15-6) \quad z = u_1 u_2 / w_3$$

the error may be estimable by the substitution $u_3 = 1/w_3$ and the application of Eq. (5.15-5).

This case occurs when the bulk velocity of a fluid in a duct is to be estimated from local measurements with a traversing pitot-static probe and a traversing temperature probe, or with rakes of such probes.

Example 5.15a. The local linear velocity v and the local density ρ of a fluid flowing in a circular pipe of radius R are given by

$$v = v_c (1 - r^2/R^2)$$

$$\rho = \rho_c (1 + \alpha r^2/R^2)$$

where r is the radius at which v and ρ occur.

The true mass flow rate through the pipe is

$$\dot{m} = \int_0^R v \rho \cdot 2\pi r dr$$

The area-averaged velocity \bar{v} and density $\bar{\rho}$ are given by

$$\pi R^2 \bar{v} = \int_0^R v \cdot 2\pi r \cdot dr$$

$$\pi R^2 \bar{\rho} = \int_0^R \rho \cdot 2\pi r \cdot dr$$

so that the fractional error in assuming

$$\dot{m} \approx \pi R^2 \bar{v} \bar{\rho} \quad [\text{Note N5.15.1}]$$

is

$$\alpha / (6 + 3\alpha) \quad .$$

If this error is to be less than 1, 2, 3, or 5 percent, α must be smaller than 0.06, 0.12, 0.19, or 0.35, respectively.

The velocity profile postulated in this example is that of fully developed laminar flow.

Example 5.15b. If the velocity and density profiles in Example 5.15a were

$$v = v_c (1 - x^n) \quad [\text{Note N5.15.2}]$$

$$\rho = \rho_c (1 + \alpha x^n)$$

where $x \equiv r^2/R^2$ and r is the radius at which v and ρ occur, the fractional error in assuming

$$\dot{m} \approx \pi R^2 \bar{v} \bar{\rho} \quad [\text{Note N5.15.3.}]$$

would be

$$[n / (2n + 1)] [\alpha / (n + 1 + \alpha)]$$

Example 5.15c. The instantaneous sinusoidal current i through an impedance element and the instantaneous sinusoidal voltage drop e across the element are given by

$$i = i_m \sin(2\pi t/T)$$

$$e = e_m \sin[(2\pi t/T) - \varphi]$$

where T is the period of the electrical oscillation. Average current, voltage, and power, as measured by integration over the time interval

$$0 \leq t \leq T/2$$

are

$$\bar{i} = 2i_m/\pi$$

$$\bar{e} = (2e_m/\pi) \cos \varphi$$

$$\overline{ei} = (1/2)i_m e_m \cos \varphi$$

Hence the fractional error in using the product of the averages rather than the average of the products is

$$(\pi^2/8) - 1 \approx 23 \% \quad .$$

The error is independent of the phase angle φ as long as the integration interval is from $t = 0$ to $t = T/2$.

Use of the product of rms averages rather than the product of arithmetic ones would yield the true average power.

Example 5.15d. If each of three quantities u_1, u_2, u_3 is expressible in terms of a normalized dimensionless parameter x in the form

$$u_j = a_j[1 + (b_j - 1)x] \quad (j = 1, 2, 3)$$

where

$$0 \leq x \leq 1; \quad b_j > 0; \quad b_j \neq 1$$

and if the individual (arithmetic) averages are defined as

$$u_{j0} = \int u_j dx$$

(all integrals are from $x = 0$ to $x = 1$),
then the fractional errors in estimating the functions

$$f_a = \int (1/u_3) dx; \quad f_b = \int (u_1/u_3) dx; \quad f_c = \int (u_1 u_2 / u_3) dx$$

by means of the approximations

$$f_a \approx 1/u_{30}; \quad f_b \approx u_{10}/u_{30}; \quad f_c \approx u_{10}u_{20}/u_{30}$$

are given by

$$\Delta f_a / f_a = (\alpha_3 / 2) / (b_3 - 1)$$

$$\Delta f_b / f_b = \alpha_3 (b_3 - b_1) / [(b_1 + 1)(b_3 - 1)^2]$$

$$\Delta f_c / f_c = 2\alpha_3 (b_1 - b_3)(b_2 - b_3) / [(b_1 + 1)(b_2 + 1)(b_3 - 1)^3]$$

where

$$\alpha_3 \equiv (b_3 + 1) \ln b_3 - 2(b_3 - 1)$$

[Note N5.15.4]

If x represents the quantity $\sin(2\pi t/T)$, the formulas apply to time integration over a quarter period ($T/4$) of a sinusoidal wave. If x represents the quantity r^2/R^2 , the formulas apply to integration over the cross section of a circular pipe of radius R .

5.16 Nonlinear averaging. If instrument indication y represents the value x of some physical variable, and if $y(x)$ is a continuous function, a periodic variation of x about some mean value x_0 will produce a corresponding mean indication y_0 . However, y_0 will not correctly represent $y(x_0)$ if

- the arithmetic mean of x is desired, but y is not a linear function of x ;
- the rms mean of x is desired, but y is not a linear function of x^2 ;
- the geometric mean of x is desired, but $\ln y$ is not a linear function of $\ln x$.

The difference between the indication $x(y_0)$ and the true mean x_0 will depend on the magnitude and wave shape of the periodic variation; if the difference is sufficiently small, it may be neglected. A consideration

of the magnitude of the difference in certain simple or extreme cases may be sufficient to provide an intuition of whether some comparable actual situation warrants consideration of nonlinear effects.

Case A. The variable of interest, x , varies sinusoidally with time, so that

$$(5.16-1) \quad x = x_0[1 + b \sin(2\pi t/T)]$$

but the measuring instrument is such that its indication y is given by

$$(5.16-2) \quad y = ax^n$$

If $b \ll 1$, if averages are taken over an integral multiple of T or over a time interval very much larger than T , and if y_0 denotes the arithmetic average of y , then,

(a) if x_0 denotes the arithmetic average of x , the fractional error in x_0 is

$$(5.16-3) \quad [x(y_0) - x_0]/x_0 \approx (n-1)b^2/4;$$

(b) if x_0 denotes the rms average of x , the fractional error in x_0^2 is

$$(5.16-4) \quad [x^2(y_0) - x_0^2]/x_0^2 \approx (n-2)b^2/4.$$

Thus,

(a) when $n = 1$, there is negligible error in measuring the arithmetic average;

(b) when $n = 2$, there is negligible error in measuring the rms average;

(c) for any n , as long as the relation (5.16-2) holds, there is negligible error in measuring the geometric average.

Case B. If x has a rectangular waveform with period T , so that, if m is an integer,

$$(5.16-5a) \quad x \equiv x(t) = x_0(1 + b) \quad \text{for} \quad mT \leq t \leq (m + 1/2)T$$

$$(5.16-5b) \quad x \equiv x(t) = x_0(1 - b) \quad \text{for} \quad (m + 1/2)T \leq t \leq (m + 1)T$$

and if the instrument is such that

$$(5.16-2) \quad y = ax^n$$

then, if y_0 denotes the arithmetic average of y over an integral number of cycles (or over a time interval that includes a very large number of cycles), the value of the fractional error

$$[x(y_0) - x_0]/x_0$$

is given by

$$(5.16-6a) \quad 2^{-(1/n)}[(1+b)^n + (1-b)^n]^{(1/n)} - 1$$

which may be approximated, for $b \leq 0.8$, by

$$(5.16-6b) \quad \begin{array}{ll} 1.43b^2/(1 + 1.3b^2) & \text{if } n = 4 \\ 0.50b^2/(1 + 0.2b^2) & \text{if } n = 2 \end{array} \quad \begin{array}{ll} -0.25b^2(1 + 0.4b^2) & \text{if } n = 1/2 \\ -0.37b^2(1 + 0.4b^2) & \text{if } n = 1/4 \end{array}$$

Example 5.16a. In an orifice-type flowmeter, the measured differential pressure varies with the square of the velocity of flow. If there is a high-frequency fluctuation in this velocity, superposed

on the mean flow velocity, the differential pressure indicated by a manometer too slow to follow the fluctuation will correspond to a velocity higher than the arithmetic mean flow velocity.

Example 5.16b. A constant-temperature hot-wire anemometer utilizes an electronic circuit to provide substantially instantaneous adjustment of wire current to maintain constant wire temperature; the wire current is measured. This instantaneous current is proportional to the fourth root of the instantaneous fluid velocity. If a rapid, periodic fluctuation is superposed on the steady flow velocity, without any flow reversal, the arithmetic mean of the current will represent a fluid velocity lower than the arithmetic mean fluid velocity.¹⁰

Example 5.16c. In a viscous fluid stream, in which there exists a constant gradient of velocity in a direction perpendicular to the flow velocity, velocity is deduced by measuring the force on a small disk ("impact disk") whose plane is normal to the flow. The force on an element of area of the disk is proportional to the square of the velocity of the fluid impinging on that element of area. The measured force will correspond to a velocity higher than that existing at the center of the disk. However, the measured force will correctly correspond to the local impact force, local impact pressure, or local rate of momentum change at the center of the disk.

Example 5.16d. A radiation pyrometer is sighted upon an area which is not of uniform temperature. Since the pyrometer's indication is proportional to the fourth power of the temperature of the surface, the temperature corresponding to the indication will be higher than the arithmetic mean temperature of the area. However, the indication will correctly yield the arithmetic mean of the radiated energy.

5.17 Energy or power transfer in measurements. The use of an instrument to measure a physical variable usually has one of two goals:

- (1) to determine the condition that exists while the instrument is installed, or
- (2) to determine the condition that would exist if the instrument were absent.

In either case, one must consider the effect that the presence of the instrument has on the physical variable being measured. In the first case, the measurement is made because such effect is considered acceptable or can be made so by adequate and convenient control of the measurement conditions. In the second case, it is necessary to establish

- (a) that the presence of the instrument produces a negligible effect, or
- (b) that the quantitative effect of the instrument's presence is known sufficiently well so that a systematic correction can be applied to the instrument's indication in order to describe conditions after the instrument has been removed (any uncertainty in the knowledge of this correction becomes a random error); or
- (c) that the measurement conditions can be adjusted, and any auxiliary measurements made, so that situation (b) can be realized.

The determination of the extent to which the presence of an instrument affects the physical variable being measured can usually be made by considering the ratio of the energy taken from the source by the instrument to the energy stored in the source; when the measurement is independent of the time interval during which the instrument is installed, the word "power" may replace the word "energy".

In practical situations, it is often adequate and more convenient to deal with some physical variable other than energy or power.

Example 5.17a. An ammeter is installed in an electrical circuit. If the ammeter is later to be removed, its resistance must be known, because the absence of this resistance may alter the current in the circuit. [Note N5.17]

¹⁰Many practical constant-temperature hot-wire anemometers utilize a linearizing circuit that yields an electrical signal proportional to the fourth root of the wire current; this signal, rather than the wire current, is measured. Thereby, the correct average of a fluctuating signal can be obtained.

Example 5.17b. A voltmeter is used to measure the voltage across a resistor. If the resistance of the resistor is not negligible compared to the voltmeter resistance, the voltage across the resistor will change after the voltmeter is removed.

Example 5.17c. The temperature of an object with low thermal capacitance (mass x specific heat) cannot be measured correctly with a thermocouple unless the wires leading from the junction lie along isotherms to prevent conduction of heat along the wires; or unless a systematic correction is made for such heat conduction.

Example 5.17d. A turbine-type or orifice-type flowmeter in a pipe introduces a flow impedance, so that the flow rate would be different in the absence of the flowmeter. At any given fluid velocity, the effect is proportional to the ratio of pressure drop across the flowmeter to the pressure differential available to produce the pipe flow. (The product of volumetric velocity and pressure represents power.) However, there is no error in flow measurement, due to this pressure drop, if the flowmeter remains in the pipe as a permanent part of the installation.

Example 5.17e. The presence of a traversing pitot tube, impact-plate flowmeter, or propeller meter to determine the distribution of local flow velocity may affect the area or the pressure differential available for the flow; thereby, the relation between local velocity and bulk velocity may be affected.

Example 5.17f. An ion gauge is attached to an enclosed high-vacuum chamber in order to measure gas density or gas pressure in that chamber. This gauge may act as a source or sink of gas, so that the gas density when the gauge is on is not the same as the gas density when the gauge is off.

If an ion gauge is used to measure the pressure in a plenum through which there is a continuous flow of gas from an external source, the error will be small if the volumetric rate of flow of this gas is high compared to the (positive or negative) volumetric pumping speed of the gauge. (The product of volumetric flow rate and pressure represents power.)¹¹

Example 5.17g. A strain gauge is attached to the surface of a thin sheet in order to measure the elongation of the sheet. If the force required to operate the strain gauge in order to produce a given elongation is not negligible compared to the force required to produce the same elongation in the sheet alone, there will be an error in the determination of the force-strain relation that would exist in the absence of the strain gauge. (The product of force and elongation represents energy.)

5.18 Power transfer in null measurements. One of the methods of realizing the condition that the presence of the instrument shall have negligible effect on the physical quantity being measured (condition (a) of Sec. 5.17) is to use a separate external source to provide most of the power that is required by the instrument to make the measurement. An effective technique is that of null measurement; it has the following important features:

- (1) It uses a detector that need not be accurate
- (2) It permits use of a detector with very high sensitivity—one whose sensibility (item (5) of Sec. 5.0) is much smaller than the probable error of measurement one is seeking to achieve
- (3) It still requires that the source being measured provide a slight amount of power (or else an off-null condition could not be detected), even though this amount is a small fraction of what would otherwise be required to effect a measurement.

The null-measurement technique is illustrated by the following case examples:

¹¹The effect of pumping by an ion gauge may be determined (and used as a systematic correction if the effect is large enough) by having a second gauge attached to the chamber or plenum, and observing the change in indication of this second gauge when the first gauge is turned on or off.

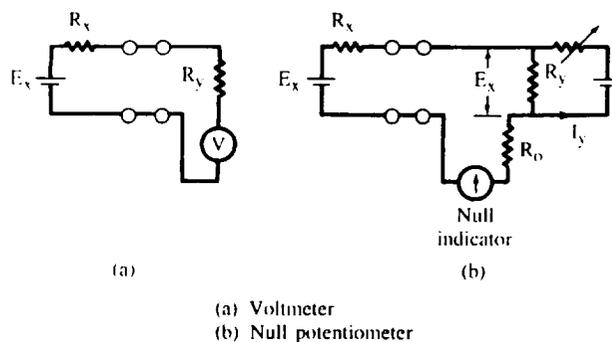


Figure 5.18.1.—Methods of voltage measurement.

Case A. Voltage measurement. In Fig. 5.18.1(a) the emf E_x generated by a source whose internal resistance is R_x is to be measured with a voltmeter whose resistance is R_y . When the voltmeter is connected, the voltmeter indication is

$$E_y = E_x R_y / (R_x + R_y)$$

rather than E_x . The error may be eliminated if the voltmeter is replaced by the potentiometer shown in Fig. 5.18.1(b). A potential difference equal to E_x is created by passing a controlled current I_y through a resistor R_y . The current is derived from an external source. Either I_y or R_y is adjusted until $I_y R_y = E_x$. This equality is indicated by zero deflection of the null indicator, whose resistance is R_0 . Both I_y and R_y are known very accurately, or else the product $I_y R_y$ is measured very accurately by comparison with an accurately known emf, like that of a standard cell. (See, also, Examples 1.4a and 1.4b). The result does not depend on R_x or R_0 or on the null-detector's calibration.¹²

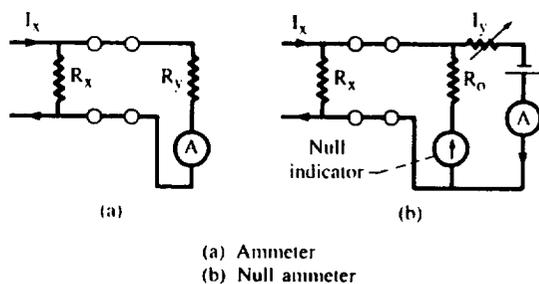


Figure 5.18.2.—Methods of current measurement.

Case B. Current measurement. In Fig. 5.18.2(a), the current I_x from a source with shunt resistance R_x is to be measured by inserting an ammeter of resistance R_y between two terminals that would otherwise be connected to each other (short-circuited). If I_x is the current that existed in this short circuit before the ammeter was inserted, the current indicated by the ammeter is

$$I_x R_x / (R_x + R_y)$$

rather than I_x . The error may be eliminated if the ammeter is replaced by the circuit shown in Fig. 5.18.2(b). A current I_y derived from an external source is adjusted until the null indicator, whose resistance is R_0 , indicates that no current is passing through it. Then $I_x = I_y$. The current I_y is measured very accurately. The result does not depend on R_x or R_0 or on the null-detector's calibration.¹³

¹²The representation of a voltage source as a series combination of an open-circuit emf and a series resistance is Thevenin's theorem.

¹³The representation of a current source in terms of a current and a shunt resistance is Norton's theorem.

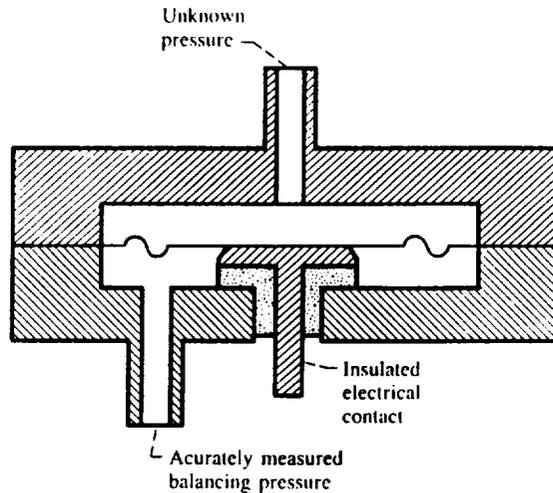


Figure 5.18.3—Null method of pressure measurement.

Case C. Pressure measurement. In Fig. 5.18.3, a flexible metallic diaphragm separates two chambers. The unknown pressure, applied to one chamber, holds the diaphragm against a mechanical stop in the other chamber. This stop is electrically insulated from the remainder of the assembly. The pressure in the chamber containing the stop is increased until electrical contact is broken; the pressure at this instant is measured accurately and is equal to the unknown pressure. The accuracy does not depend on the elastic qualities of the diaphragm. [Notes N5.18.1 and N5.18.2]

Case D. Temperature measurement. The correct value of the temperature T_x of a calorimeter is achieved only if there is no heat exchange between the calorimeter and its surroundings. This condition is achieved by surrounding the calorimeter with an isothermal enclosure whose temperature is separately controlled to be equal to that of the calorimeter. The temperature difference may be measured with a differential thermocouple of fine wires. When this difference is zero, the temperature of the enclosure may be measured with an accurate thermometer, however bulky. The measurement does not depend on the quality of the differential thermocouple, or on the emittance of the surfaces involved.

In similar fashion, a bolometer element designed to measure radiant energy may be surrounded by a separately heated isothermal enclosure whose temperature is kept equal to that of the bolometer.

In all null measurements, the relation between off-null indication y and the physical variable of interest x can be established only by drawing some power from the source, even though this power is only a small fraction of the power that would have been drawn in a non-null measurement. The reason for this requirement is that the knowledge of the existence of a balanced or null condition requires the ability to detect when an unbalance exists. Thus, if $(\Delta x)_{max}$ represents the limit of error that is acceptable in the measurement of x and if $(\Delta y)_{min}$ represents the sensibility of off-null indication (the smallest off-null magnitude that can be reliably detected), it is necessary that, at and near balance,

$$(5.18-1) \quad (\Delta y)_{min} \leq (\Delta x)_{max} .$$

The derivative dy/dx is the sensitivity of off-null detection. This derivative is a maximum when, at an unbalance Δx , the power developed in the source is equal to the power developed in the detector. In electrical circuits, this equality is achieved when certain relations exist between source impedance and null-indicator

circuit impedance. In mechanical or thermal situations, it may be more difficult to define the requirements explicitly, but Eq. (5.18-1) remains dominant.

The cases just treated will serve as examples.

Cases A and B. For a given unbalance Δx (in voltage or current to be measured), maximum voltage will appear at the detector terminals if R_0 is as high as possible; maximum current will pass through the detector if R_0 is as low as possible; maximum power will be delivered to the detector if

- (i) in a dc circuit, $R_0 = R_x + R_y$
- (ii) in an ac circuit, where a resistance R is replaced by an impedance

$$z = R + jX,$$

it is necessary that

$$R_0 = R_x + R_y$$

and that

$$X_0 = - (X_x + X_y)$$

In ac circuits, the matching of source and detector impedances is facilitated by the use of transformers. The use of a transformer that provides maximum power transfer will yield the best sensibility of off-null detection.

It should be noted that, when maximum power transfer is desired, no serious loss of power results when there is a moderate deviation from unity in the ratio

$$R_0 / (R_x + R_y) .$$

For a ratio of 2:1 or 3:1, the power delivered to the detector is only 12 percent or 30 percent less, respectively, than the maximum possible value.¹⁴

Case C. It is necessary that the diaphragm be so flexible that the pressure differential required to produce a clear separation between contacts is only a small fraction of the desired sensibility of pressure measurement. At the same time, the mechanical design must be such that the diaphragm is not damaged when full-scale pressure acts to hold the diaphragm against the stop.

Case D. Not only must the sensibility of off-null detection be much smaller than the desired sensibility of temperature measurement, but the uniformity of temperature of the surface of the enclosure must also be adequate; auxiliary differential-temperature measurements may be used to confirm this adequacy or to provide a closer estimate of the mean enclosure temperature.

5.19 Ultimate sensibility of measurements. Noise. The smallest change that is reliably detectable in any observation or measurement can be represented as the rms value of a random error in that measurement. For psychophysical observations, this value is determined by the abilities of a human observer; quantitative descriptions, based on statistical analysis of the abilities of a representative sample of persons, can be presented for an average observer, sometimes designated as the "standard observer", although the sample is often chosen from a group of physically fit young persons.

For measurements of physical variables, the limit of sensibility is determined by the randomness of the motion of molecules, atoms, or charged particles or of the randomness of occurrence of events involving these particles. The magnitude of sensibility can be reduced by averaging the observation over a longer period of time; the magnitude usually is reduced in proportion to the square root of the averaging-time interval Δt .

¹⁴This result derives from the properties of the maximum or minimum: a large change in the abscissa produces only a small change in the ordinate. This effect entered also into the discussions of Secs. 3.28 and 4.10.

In electrical measurements, it has become customary to consider that the quantity has a frequency spectrum and that averaging is the process of measurement over a frequency interval Δf . This quantity is on the order of magnitude of $1/(2\Delta t)$ but not necessarily equal to it. The advantage of using Δf is that many steady-state electrical measurements represent the average (usually rms average) value of a signal that is fluctuating at a frequency near to f ; Δf is then an interval centered around f .¹⁵

The term *noise* has been affixed to the rms value of the random fluctuations that ultimately limit the ability to make a measurement. When a detector is used to measure the output of a source of energy, both source and detector individually contribute noise. Furthermore, there may be several origins of noise in both source and detector. Each origin of noise may contribute a separately definable magnitude of noise. The resultant sensibility, that constitutes the limit of reliable detection, is the square root of the sum of the squares of all of these magnitudes, provided each of these contributions has the same dimensions.

Some common limits of sensibility are listed below. (When they appear in the formulas given in this section, T is absolute temperature and k is Boltzmann's constant.)¹⁶

A. Psychophysical optical observations.

(1) The ability to detect the coincidence of two index lines is termed the *vernier acuity* of the eye. It is of interest in optical range finding and in dimension measurement by optical projection. It is on the order of 12 seconds of arc for a person with excellent vision. For the unaided eye at normal viewing distance, the vernier acuity becomes 0.015 mm; this distance is reduced in proportion to any linear magnification that is used.

(2) The ability of the eye to detect the separation between two lines is termed the *visual acuity* of the eye. It is on the order of 1 minute of arc for a person with excellent vision. For the unaided eye at normal viewing distance, the visual acuity becomes 0.07 mm; it is reduced in proportion to any linear magnification that is used.

(3) The ability of the unaided eye to detect the presence of an appropriately illuminated scratch on a polished surface is due to diffraction by the scratch. A 1 μm scratch, subtending a viewing angle of 1 second, is detectable. The observation provides information only on scratch presence—not on scratch width.

(4) The eye can discriminate between two levels of luminance that differ by 2 percent, if the eye has been fully adapted to the luminance level.

For example, application of this fact to the manually balanced disappearing-filament optical pyrometer indicates an ability to achieve balance to 1 K at a temperature of 1000 K.

B. Mechanical measurements. The ability to make mechanical measurements of position is limited by Brownian motion and is rarely of interest in industrial or routine laboratory applications, but does enter into some highly sensitive measurements in research or standards laboratories.

(1) When particles are suspended in a stagnant liquid, and a particle's position is averaged over a time interval Δt , the rms uncertainty δx in position x is given by

$$(5.19-1) \quad (\delta x)^2 = 2D \cdot \Delta t$$

where D is the diffusion coefficient of the liquid. [Note N5.19.1]

(2) When the linear deflection x of a spring, whose spring constant is K_x (force per unit deflection), is to be determined, the rms value of the uncertainty δx is given by

$$(5.19-2) \quad (\delta x)^2 = kT / (2K_x) \quad .$$

(3) When the rotation φ of a torsion pendulum, whose torsional spring constant is K_φ (torque per unit angle of twist), is to be determined, the rms value of the uncertainty $\delta\varphi$ is given by

¹⁵The precise definition of Δf must be made by the user; a common definition uses the "bandwidth at half maximum" of a bandpass filter. (See Sec. 7.20.)

¹⁶At 300 K, $kT = 4 \times 10^{-21}\text{J}$.

$$(5.19-3) \quad (\delta\varphi)^2 = kT/(2K_e)$$

C. Measurement of radiant energy.

(1) The rms value δP of the random fluctuations of total radiant power P emitted by a radiator of area A and total emittance e is given by

$$(5.19-4) \quad (\delta P)^2 = 8eA\sigma kT^5 \cdot \Delta f$$

where σ is the Stefan-Boltzmann constant. (Ref. 5-1, p. 59)

(2) The rms value δT of the random fluctuations of temperature T of a body, like a radiation receiver, whose heat capacity (thermal energy required to produce unit temperature rise of the body) is C_b , is given by

$$(5.19-5a) \quad (\delta T)^2 = kT^2/C_b \quad . \quad (\text{Ref. 5-1, p. 354})$$

The rate at which the body can exchange energy with its surroundings is proportional to its heat transfer rate K_b (thermal power it exchanges per unit temperature difference). The rms value δP of the fluctuation in this power, because of the fluctuation δT , is given by

$$(5.19-5b) \quad (\delta P)^2 = (K_b \cdot \delta T)^2 = 4kK_b T^2 \cdot \Delta f \quad . \quad (\text{Ref. 5-1, p. 356})$$

(3) When the values of δP for both emitter and receiver are of significant magnitudes, the values of $(\delta P)^2$ given by Eqs. (5.19-4) and (5.19-5b) must be summed, after appropriate weighting for the solid angle subtended by the receiver, to obtain the total noise due to these causes.

(4) For a laser operating as a single-frequency oscillator far above threshold, the rms variation δn in the number n of quanta emitted in any given time is \sqrt{n} . (Ref. 5-2, p. 105)

D. Measurement of electrical quantities.

(1) *Thermal noise* (Johnson noise, Nyquist noise) is usually the principal type of noise found in electrical components. [Note N5.19.2] It appears as an emf across the terminals of a resistive element even when the element is open-circuited and even when no current is passing through the element. When thermal equilibrium exists, the rms value δe of the random fluctuations in the emf e that appears at the open-circuited terminals of a resistance R is given by

$$(5.19-6a) \quad (\delta e)^2 = 4kTR \cdot \Delta f \quad .$$

(At 300 K, the noise power is $1.7 \times 10^{-20} \cdot \Delta f$ watt if Δf is in hertz.)¹⁷

(a) If the resistor is connected to a capacitance C and if the frequency f around which the interval Δf is centered is such that

$$(5.19-6b) \quad f \gg 1/(2\pi RC)$$

then

$$(5.19-6c) \quad (\delta e)^2 = kT/C$$

because the capacitor acts as an integrator of the noise. (At 300 K, the noise energy is 2×10^{-21} J.)

(b) If the resistor is connected to a critically damped galvanometer (Sec. 6.3) whose natural frequency is f_n , then the integration of noise is such that

¹⁷The measurement of noise power in a wire resistor has been proposed as a means of temperature measurement (the *noise thermometer*) because the power generated depends only on the temperature and on Δf .

$$(5.19-6d) \quad (\delta e)^2 = \pi kTRf_n/2 \quad .$$

(2) *Current noise* represents a random fluctuation δi in the current i passing through an electrical component. The magnitude of the fluctuation is proportional to some power of the current i . There are several possible sources, distinguished by their nature or their behavior.

- (i) Shot noise [Note N5.19.3]
- (ii) Generation-recombination noise (G-R noise) [Note N5.19.4]
- (iii) Flicker noise ($1/f$ noise) is such that noise power varies inversely with frequency. Its cause is not well understood; its magnitude appears to depend on surface conditions.

(3) When several sources contribute noise, the total noise voltage or current that appears at any pair of terminals is the square root of the sum of the squares of the individual noise voltages or currents. This quantity is sometimes described as the *root sum square* (rss) voltage or current. (The summation may be made only of quantities having the same dimensions.) The total noise power generated is merely the sum of the individual noise powers.

Example 5.19. If, in order to achieve maximum power transfer (Sec. 5.18), a load resistor at temperature T_2 is connected to a source resistor of equal value but at temperature T_1 , then the noise power in each resistor is $k(T_1 + T_2) \cdot \Delta f$. (If $T_1 = T_2 = 300$ K, the noise power in each resistor is $0.8 \times 10^{-20} \cdot \Delta f$ watt)

(4) When a device has more than two terminals or when there is a chain of devices, it is convenient to state the *equivalent input noise*—the noise at the input terminals that would create the same noise at a designated pair of output terminals if all succeeding components of the chain were noise free. This equivalent input noise may be described in one of several ways:

- (i) by stating the *equivalent input noise power*, as described above
- (ii) by stating the *equivalent noise resistance* R_n , the value of a resistor whose thermal noise emf, at the temperature T of the sources (if all sources are at the same temperature) or at an arbitrarily specified temperature T , is the same as the total noise emf

$$(5.19-7) \quad R_n = [\text{total } (\delta e)^2] / (4kT \cdot \Delta f)$$

The power fluctuation δP associated with a voltage fluctuation δe across a resistor R_n or with a fluctuation δi in the current through that resistor is given by

$$(5.19-8) \quad \delta P = (\delta e)^2 / R_n = (\delta i)^2 \cdot R_n \quad .$$

- (iii) by stating the value of the equivalent input δe or δi
- (iv) by stating the *equivalent noise temperature*, T_n , the temperature of a resistor whose thermal noise power would be equal to the total noise power

$$(5.19-9) \quad T_n = (\text{total noise power}) / (4k \cdot \Delta f)$$

- (v) by stating the *noise factor*, F , the ratio of the noise power that actually appears at the designated output terminals of the device, to the noise power that would appear at the output if the only noise source were an ideal (noninductive and noncapacitive) wire resistor whose value was equal to the resistance of the source that is connected to the input terminals of the device. Thus, one may speak of the noise factor of a semiconductor, of an amplifier, or of a semiconductor connected to an amplifier.

$$(5.19-10) \quad F = \frac{\text{actual noise power output}}{\text{noise power output from ideal source}}$$

This ratio may be expressed in decibels as $(10 \log F)$ dB.

The terms *noise factor* and *noise figure* have been used by some writers to denote the power ratio itself or its value in decibels, but there has been no consistency in the allocation of names. Current U.S. practice is to use the terms interchangeably, regardless of what (nondimensional) unit is used. A further standard practice is to specify that, when F is used in a product specification to characterize the performance of a device, the temperature of the ideal resistor shall be presumed to be 290 K, and the bandwidth shall be presumed to be 1 Hz.

Note N5.19.5 gives some empirical information on noise sources and magnitudes.

5.20 Signal-to-noise ratio. The indication of an instrument intended to measure some physical variable is impaired by the presence of random errors in that indication. The indication that would be obtained in the absence of such errors may be termed the “signal”; the indication that would be obtained in the presence of only the random errors may be termed the “noise”. An effective way to determine whether noise introduces a significant uncertainty in the measurement of a signal is to compare the signal power S (the power available, at the point of measurement, from the source being measured) with the noise power N available at the same point of measurement. The signal-to-noise power ratio (S/N) is sometimes stated in dB (Eq. 2.8-1a).

The noise power N may be considered to originate from the sources treated in Sec. 5.19, from random fluctuations in the variable being measured, or from randomly fluctuating external physical variables that produce corresponding fluctuations in the signal at the point of measurement. On occasion, systematic fluctuations have been included in the definition of noise as a matter of convenience; for example, power-line hum when its evaluation, correction, or cancellation is inconvenient. On such occasions, it is necessary that an author define what is included in N .

There remains the question of how acceptable is a given ratio S/N . The distributions of the numerous random events that cumulatively contribute to the noise are sufficiently Gaussian, and the central limit theorem (Sec. 3.29) is sufficiently effective so that the Gaussian distribution (Sec. 3.18) may be considered to apply to the noise. The total rms noise amplitude (voltage, current, displacement, etc.) then corresponds to the quantity σ in Eq. (3.18-1) and to the abscissa e_σ in Fig. 3.18(a).

Loosely speaking, one may say that, for noise with a Gaussian amplitude distribution, if the signal amplitude is 2 or 3 times the noise amplitude ($S/N = 4$ or 9, respectively), the odds are about 1:20 or 1:400, respectively, that the signal will be obscured by the noise (Ref. 5-3, p. 22).¹⁸

5.21 Improvement of signal-to-noise ratio. If N represents only random noise, the value of S/N may be increased by increasing the time interval Δt in which a reading is averaged or by narrowing the frequency band Δf within which a signal is being accepted. Mechanical means of doing this are use of an instrument with low natural frequency or long time constant (Chapter 6). Electrical means are

- (a) use of analog integration
- (b) use of an integrating digital meter, such as one that converts voltage or current to pulse frequency, so that the pulses may be counted over a long period of time
- (c) use of synchronous modulation and demodulation (e.g., with electromechanical, photoelectric, or solid-state-circuit choppers) with a long-time-constant detector; if the original noise lay in a frequency interval Δf and the detector time constant or the integration time is τ , the factor by which the original signal-to-noise ratio is improved is on the order of $\sqrt{(\tau \Delta f)}$
- (d) use of narrow-bandpass filters.

The choice of an averaging-time interval or its equivalent always represents a balance between the amount of noise reduction and the ability to follow fluctuations in the variable being measured.

Other methods of increasing the signal-to-noise ratio are

- (a) reduction of the temperature of the principal sources of noise that are temperature dependent [Note N5.21],
- (b) use of null-potentiometer techniques to reduce current in a source of noise that is current dependent.

¹⁸Note that Ref. 5-3 uses S/N to denote the square root of the power ratio.

5.22 Instrument standards. In any one laboratory, instruments in daily use are occasionally checked by comparison with "working standards;" these, in turn, are occasionally checked by comparison with instruments which are "primary standards" for that laboratory. The primary standards may occasionally be checked by comparison with standards maintained by a national or regional standards organization, but such comparison may be rare and is usually inconvenient because great care may be required in transportation of the laboratory's primary standard to another location, and considerable recovery and stabilization time may be required after such transportation before a reliable comparison can be made.

The laboratory's own primary standards should be maintained under conditions that ensure virtual certainty that they have remained unchanged since the comparison with the national or regional standards laboratory. Some means for such assurance are

(1) The standards are in a room in which temperature and humidity are controlled, in which vibration and acoustic noise are minimal, and in which temperature and humidity are unlikely to change appreciably when there is a temporary failure of electric power or other utility service. A windowless basement room, well away from noisy, vibrating machinery, is often a satisfactory location.

(2) Personnel who use these standards are few in number, permanent, professionally trained, and temperamentally suited. The professional training must include an understanding of the instruments and the techniques of using them, the ability to recognize and correct for systematic errors, and the ability to recognize and estimate random ones. Temperamentally, the personnel must be willing to attend carefully to all the numerous details usually required, and must have the integrity to admit any inadvertent mistake that may have compromised the accuracy of a standard; at the same time, they must have been assured, in advance, that such inadvertence would not be penalized.

(3) The packaging and transportation of the laboratory's primary standard to and from the national or regional standards laboratory must be performed with sufficient care to ensure that safe limits of temperature, acceleration, and vibration have not been exceeded.

5.23 Criteria for instrument selection. The selection of an instrument for a particular application requires a human judgment that is based on the needs of that application and the characteristics of the available instrumentation. Most of the instrument characteristics can usually be expressed in quantitative form; fewer of the needs can usually be so expressed. Once the human judgment is made, it may be possible to formulate the conclusion as the sum of a number of weighted numerical gauges of each item considered. The items and the weights will vary with each application. Any formula for selection is the result of a human judgment; it is not a substitute for such judgment.

The magnitude of systematic and random errors of an instrument, as treated in this chapter and further treated in following chapters, will depend on the range of circumstances under which the instrument will be expected to operate satisfactorily. The weight (importance) assigned to these errors will depend on the measurement accuracy required and on other performance factors, like those outlined below. In gauging the merits of any instrument for any given application, caution must be exercised not to condemn it over another merely because more is known of its errors than is known of the errors of the other, nor to appraise one instrument highly merely because less is known of its errors than is known of the errors of others.

One of the more common compromises required is between speed and accuracy. The subject of dynamic response will be treated in the chapters that follow, but even when a so-called static measurement is made, the time required to make the measurement usually must not be excessive, even though random errors are always reduced when the integration interval is increased.

Some other significant criteria for selection of an instrument or a measurement technique are listed here.

(1) Installation and integration: quality and cost of personnel required to make the initial installation and to integrate it into the complete system or operation.

(2) Maintenance required: frequency, quality, and cost of the work; quality, permanence, and cost of the personnel required and of their instruction.

(3) Reliability: probability, severity, and consequences of malfunction.

(4) Serviceability: availability of parts and of repair service over the life of the instrument; quality and cost of personnel required for such service. (The cost of available parts is usually a trivial fraction of other costs.)

[Note N5.23]

NOTES FOR CHAPTER 5

N5.1. In this example, or whenever a pointer-and-scale instrument is used to provide a quick indication of a momentary condition (like the maximum value of a momentary pulse), it is important that the indication be quickly perceptible. A broad, spade-type pointer and a coarsely graduated open scale (Fig. 5.1) that relies on visual interpolation between scale divisions is more useful than a finely divided scale or a knife-edge pointer.



Figure 5.1.—Pointer and scale design for quick perception.

N5.4 The deviation of the sea level value from the standard value g_n (Table 2-1) is less than 0.3 percent, as reported in the Smithsonian Physical Tables for over 100 locations, worldwide. The reduction with altitude is 0.03 percent per kilometer.

N5.6.1 If separation of these effects is of interest, the following procedures may be helpful:

- (i) when backlash is negligible, elastic hysteresis in the presence of dry friction can be evaluated by performing calibrations in the presence of mild vibration. In an electrobalance, dither (a superposed alternating current of relatively small amplitude) may be used.
- (ii) when dry friction is negligible, elastic hysteresis in the presence of backlash can be evaluated by performing a calibration in one direction without ever overshooting a calibration point and then performing a calibration in the opposite direction by overshooting each calibration point and then returning to the desired input. The amount of overshoot should be slightly larger than the backlash.
- (iii) when both backlash and friction are significant, they generally mask the elastic hysteresis.

N5.6.2 The magnitude of the hysteresis and drift is approximately proportional to the magnitude of the change in spring deflection that produced it. The values of hysteresis and drift will therefore be largest when the input change represents the full span of the instrument.

N5.6.3 The drift generally occurs exponentially with time, so that the “final” indication is the asymptote of the curve of indication vs time.

N5.6.4 The temperature coefficient of the modulus of elasticity of NiSpan C, a high-quality spring material, is about $-2 \times 10^{-5}/^\circ\text{C}$ (it varies with the heat-treatment temperature); that of ordinary high-quality spring steel is about $-20 \times 10^{-5}/^\circ\text{C}$. The coefficient of linear thermal expansion of both materials is about $1 \times 10^{-5}/^\circ\text{C}$.

N5.7.1 The net thermal emf to be expected at the terminals of an instrument are

- 3 to 10 μV in high-quality industrial instruments
- 1 to 3 μV in high-quality instruments for routine laboratory use
- 0.1 to 1 μV in highest-quality laboratory standard instruments

N5.7.2 In some applications, manipulative techniques may permit reducing the effect of thermal emf's by a factor of 10 or 100. For example, in using a Wheatstone bridge, reversal of power-supply polarity, leaving all other wiring untouched, permits averaging of the null-detector indications to yield the correct off-null reading.

- In other applications, one may seek to create an isothermal environment for the low-level circuitry by
- (i) mounting positive and negative terminals next to each other on a massive copper block with thin insulating bushings;
 - (ii) surrounding the low-level circuitry with an isothermal enclosure;
 - (iii) minimizing the amount of heat generated in the enclosure;
 - (iv) using only massive all-copper switches, copper wire and terminals, manganin resistors, and low thermal emf solder.

N5.7.3 The electric and magnetic effects of power-line hum may be reduced by some of the following expedients:

- (i) use of coaxial or triaxial cables and electrically insulated enclosures, whose shields are all connected together at one and only one terminal
- (ii) connecting this terminal to a point at the same ac potential as the ground or neutral line of the power supply
- (iii) filtering the power-line input to the measurement system to remove significant harmonics
- (iv) if hum is of steady amplitude and phase, using a hum-bucking circuit that provides independent control of amplitude and phase
- (v) twisting of wire pairs that carry the same signal in opposite directions
- (vi) shielding these pairs with material of high magnetic permeability
- (vii) using power supply transformers of balanced construction, with separately shielded primary and secondary windings (Interwinding capacitance is usually less than 1 pF.)
- (viii) physical separation of low-level circuits from the power lines
- (ix) use of band-rejection filters, preceding the measuring instrument
- (x) measuring the average value of the signal over a sequence of time intervals, each of which is exactly equal to the period of the hum, or to an integral number of such periods. The effect of hum is eliminated and higher-frequency noise is attenuated. An average of these averages will attenuate noise at frequencies below the hum frequency. (The successive intervals need not be contiguous, so that switching operations may be interposed between these intervals.) Integrating digital voltmeters often use this method of hum rejection.

N5.10.1 A laminar profile is to be expected if the Reynolds number N_{Re} of the flow is less than 1000. The profile is then given by

$$u/u_c = 1 - r^2/R^2$$

where u_c is the velocity at the center of the pipe ($r = 0$).

If a flat profile exists at the entrance of a square-ended pipe, the profile can be considered fully developed at a distance

$$x = 0.2 RN_{Re}$$

downstream from the entrance of the pipe.

N5.10.2 A turbulent profile is to be expected if the Reynolds number N_{Re} of the flow is greater than 2000. The profile is then given by

$$u/u_c = (1 - r/R)^{1/7}$$

where u_c is the velocity at the center of the pipe ($r = 0$).

If a flat profile exists at the entrance of a square-ended pipe, the profile can be considered fully developed at a distance

$$x = 40RN_{Re}$$

downstream from the entrance of the pipe.

N5.13.1. The tabulation of x -values in Table 5-1 may be used to determine the equivalent values for any other coordinate system, by the following procedure.

- (1) Let the limits of the new coordinate s be s_a and s_b so that

$$s_a \leq s \leq s_b$$

- (2) Write the element of area dA in the new coordinate system as

$$dA = B \cdot ds$$

- (3) Then the transformation between s and x is given by

$$x = \int_{s_a}^s B \cdot ds / \int_{s_a}^{s_b} B \cdot ds$$

Example 5.13a. In polar coordinates, if

$$R_a \leq r \leq R_b$$

$$dA = 2\pi r \cdot dr,$$

then

$$x = (r^2 - R_a^2) / (R_b^2 - R_a^2)$$

Example 5.13b. Fluid flowing in a pipe immediately downstream of an elbow may have a strong velocity gradient along a diameter lying in the plane of the elbow. To find bulk velocity, it may be best to take the element of area, as it is shown in Fig. 5.13, at a distance r from the pipe's center, so that

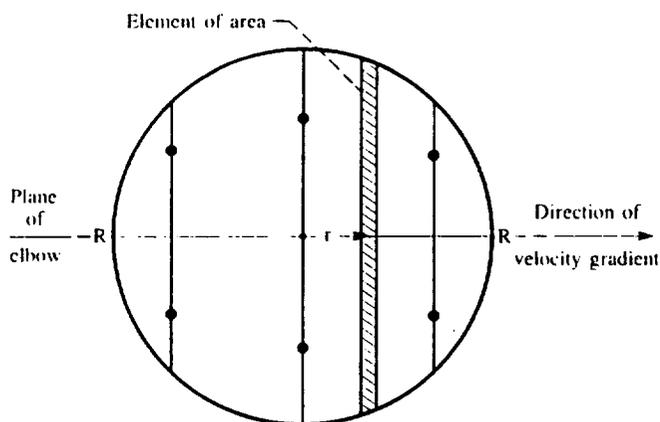


Figure 5.13.—Element of area in a circular pipe with diametral velocity gradient. Dots show locations for six-point integration.

$$-R \leq r \leq R$$

$$dA = 2dr \sqrt{R^2 - r^2}$$

Then

$$x = (1/2) + (1/\pi) \left\{ (r/R) \sqrt{1 - (r^2/R^2)} + \sin^{-1}(r/R) \right\}$$

Here, $x = 0$ when $r = -R$ and $x = 1$ when $r = R$.

However, if only a single measurement is made for each element of area, with the points of measurement lying on the pipe diameter, the computed bulk velocity will be too high. A closer approximation will be obtained if each element of area is assigned the average of two measurements made at locations which are on each side of the diametral line, by a distance equal to $0.6 \sqrt{R^2 - r^2}$ (the 0.2-0.8 rule). These locations are shown by dots in Fig. 5.13 for the case $n = 3$ and Chebyshev integration.

N5.13.2 In the most general case where u varies along three orthogonal directions X_1, X_2, X_3 , a different method of integration and a different number of locations n_1, n_2, n_3 , respectively, may be selected for each of the directions, to suit the circumstances.

Usually, methods using equally weighted measurements are preferred because then the simple sum of all measurements conveniently provides the desired integral of u .

Example 5.13c. To measure bulk fluid velocity in a circular duct, the centroid-of-equal-areas method may be used with n_1 equal to 2, 3, or 4, for measurements around a circle at any given radius; and the Chebyshev method may be used with n_2 equal to 2, 3, or 4 for measurements along any given radius. If almost axially symmetric flow is expected, $n_1 = 2$ may be sufficient. If fully developed flow is expected, $n_2 = 2$ may be sufficient. If the flow is expected to be fully developed and almost axially symmetric, a total of four measurements may be adequate.

Example 5.13d. The average speed \bar{v} of a river is measured by four vertical arrays of current meters. The arrays are in a straight line normal to the direction of flow and, by the Chebyshev method, are located at 0.1027, 0.4062, 0.5938, and 0.8973 of the river width; each array consists of two current meters located at a distance below the water surface equal to 0.2113 and 0.7887 of the local water depth. If this depth, at the four horizontal locations, is 4, 7, 8, and 5 m, respectively, and if the sum of each pair of current-meter readings is $s_1, s_2, s_3,$ and s_4 respectively, then the velocity \bar{v} is given by

$$48\bar{v} = 4s_1 + 7s_2 + 8s_3 + 5s_4$$

N5.15.1 The integrals in Example 5.15a are

$$\dot{m} = \pi R^2 v_c \rho_c (3 + \alpha) / 6$$

$$\bar{v} = v_c / 2$$

$$\bar{\rho} = \rho_c (2 + \alpha) / 2$$

N5.15.2 The progressive change in velocity profile as n decreases from an initially high value to an ultimate value of unity qualitatively resembles the development of a laminar-flow velocity profile when a fluid enters

a square-ended circular pipe from a substantially infinite plenum. The profile is initially almost square; its fully developed shape is represented by $n = 1$. For further details, see Notes N5.10.1 and N5.10.2.

N5.15.3 The integrals in Example 5.15b are

$$\dot{m} = \pi R^2 v_c \rho_c [n/(n+1)][1 + \alpha/(2n+1)]$$

$$\bar{v} = v_c \bullet n/(n+1)$$

$$\bar{\rho} = \rho_c (n+1 + \alpha)/(n+1)$$

N5.15.4 A common example in gas dynamics is the measurement of the mass flow rate through a restriction (such as the nozzle supplying air to a jet engine under test). The square of this flow rate is proportional to $p \bullet \Delta p / T$, where p is upstream pressure, Δp is the pressure drop across the restriction, and T is upstream temperature. Then $u_1 = p$, $u_2 = \Delta p$, $u_3 = T$. For practical reasons, one may choose to average separately the local temperature, or the local density (which is proportional to p/T), or the local mass flow rate.

N5.17 If $R_S =$ source resistance

$R_L =$ load resistance

$R_A =$ ammeter resistance

$E_p =$ acceptable probable fractional error of measurement

then R_A should be much smaller than $(R_S + R_L)/E_p$ unless a systematic correction is made.

N5.18.1 The method is useful when

- (a) the unknown pressure is that of a fluid to which a high-precision pressure gauge may not be safely connected.
- (b) there are many pressures to be measured. Each pressure source is connected to its own respective chamber-diaphragm assembly, but all chambers containing stops are connected to a common source and to a single high-precision pressure gauge. As pressure of the common source is increased at an accurately known linear rate, the time at which any individual contact is broken is a measure of the respective pressure.

If any individual pressure is fluctuating, it must be connected to the chamber-diaphragm assembly through a capillary tube or other pneumatic resistance in order to provide appropriate averaging of the pressure (see Secs. 6.6 and 7.12); otherwise, contact might be broken at a momentary extremum of the fluctuation.

N5.18.2 A comparable mechanical arrangement has been used for acceleration measurement; the end of an acceleration-sensitive spring-mass combination (Sec. 6.1) is held against a stop by a calibrated spring.

N5.19.1 This relation was deduced by A. Einstein in 1905. The diffusion coefficient is the mass of particles that diffuse through unit area in unit time per unit concentration gradient. (Concentration represents the mass of particles in unit volume.)

N5.19.2 Thermal noise represents random fluctuations in the velocity of free electrons in a resistive element and results principally from collisions between the free electrons and the lattice atoms.

N5.19.3 Shot noise represents fluctuations in the number of free electrons.

N5.19.4 Generation-recombination noise represents fluctuations in the number of charge carriers and is said to result from the generation and recombination of carriers, like electrons and holes in semiconductors, although a more precise description is that the noise results from randomly occurring transitions between energy levels.

N5.19.5 In the following formulas, e , i , R , g , and Δf are, respectively, in volts, amperes, ohms, siemens, and hertz. (Refs. 5-2 and 5-3; also see Footnote 18 on p. 123)

- (1) Shot noise in composition resistors (carbon, metallized film, or cermets) can be represented by a noise voltage δe across the resistor given by

$$(N5.19-1) \quad |\delta e| = c_1 \times 10^{-8} e \log^2 R$$

where

$$0.1 \leq c_1 \leq 10 .$$

- (2) For a space-charge-limited thermionic diode, the random fluctuation δi in current is given by

$$(N5.19-2) \quad (\delta i)^2 = 3(1 - \pi/4)kTg \cdot \Delta f$$

where

$$(N5.19-3) \quad g = \partial i / \partial e$$

is the transconductance at the operating point.

- (3) For a forward-biased semiconductor diode carrying small currents, the random voltage δe that appears across the diode is given by

$$(N5.19-4) \quad (\delta e)^2 = 2kT \cdot \Delta f / g$$

where g is the operating-point transconductance given by Eq. (N5.19-3).

The noise is less than the thermal noise that might be expected from Eq. (5.19-6a) because a biased diode is not a system in thermal equilibrium.

- (4) For diodes in other than the ordinary operating region (e.g., for diodes in the reverse-bias or saturated region) the noise is so much greater in magnitude that the device is not used to detect small electrical signals. In fact, a reverse-biased diode is often used as a random-noise generator.
- (5) For a thermionic triode, R_n as defined by Eq. (5.19-7) is given by

$$(N5.19-5) \quad R_n \approx 2.5/g$$

where g is the transconductance given by Eq. (N5.19-3).

Tetrodes and pentodes are about three times noisier than triodes.

- (6) For a bipolar transistor, each lead may be considered to have current noise in addition to any thermal noise developed in resistors that are in those leads. At moderate frequencies, each lead may be considered to have a random fluctuation δi due to shot noise and given by

$$(N5.19-6) \quad (\delta i)^2 = 2q_e i \cdot \Delta f$$

where i is the current in the lead and q_e is the electron's charge ($\approx 1.6 \times 10^{-19} \text{C}$)

There may also be appreciable thermal noise (Eq. 3.19-6a) generated by the base resistance. At low frequencies, this noise may be augmented by flicker noise, so that the fluctuation δi_b in base current i_b is given by

$$(N5.19-7) \quad (\delta i_b)^2 = 2q_e i_b \cdot \Delta f (1 + f_0/f)$$

where f_0 is usually on the order of 1 kHz.

- (7) For a field-effect transistor (FET), the principal effect of the combination of thermal and flicker noise can be described by stating that a random voltage fluctuation δe may be considered to have been injected into the gate lead, where

$$(N5.19-8) \quad (\delta e)^2 = 8kT(1 + f_0/f) \bullet \Delta f / (3g_m) \quad .$$

The transconductance g_m is given by

$$(N5.19-9) \quad g_m = \partial i_d / \partial e_{gs}$$

where i_d is drain current and e_{gs} is the gate potential relative to the source. The value of f_0 is usually on the order of 25 kHz but may be 10 or 100 times lower if source impedance is very high (several megohms).

- (8) For a junction field-effect transistor (JFET), there may also be current noise in the gate lead. The combined effect of current noise in all leads may be represented by an equivalent random noise current δi_g in the gate lead given by

$$(N5.19-10) \quad (\delta i_g)^2 = [(\pi^2 f^2 C^2 / g_m) + 2q_e i_g] \bullet \Delta f$$

where C is the capacitance between gate and drain when the latter is connected to the source, q_e is the electron's charge, and g_m is given by Eq. (N5.19-9).

- (9) For an insulated-gate field effect transistor (IGFET), the value of f_0 in Eq. (N5.19-8) is usually on the order of $1.5 \times 10^8 g_m \Omega \bullet \text{Hz}$; at low frequencies, the corresponding flicker noise is usually unacceptably large.

N5.21 Low-level radiation detectors used in astrophysical observations are often cooled to very low temperatures. The cooling often extends to the first amplifier stage in the chain of signal processing.

N5.23 As an example of the range of considerations that may affect instrument selection, consider a problem of controlling the mass flow rate \dot{m} of fluid in a pipe; the control of \dot{m} , part of an industrial process, is to be made without recourse to auxiliary measurements and computations. At the simplest extreme, if the composition and density of the fluid will remain substantially constant, if the piping system is fixed, and if velocity profiles, turbulence, and swirl will remain almost invariant, the control of the pressure differential Δp between two appropriate locations in the piping system may be sufficient to permit maintenance of a desired flow rate, once the correlation has been established between Δp and either \dot{m} or a final process variable that is uniquely correlated with \dot{m} . At an intermediate level of complexity, if fluid density is always known, a volume-flow meter may be used that may prove to be less expensive, or easier to install, to maintain, or to service, than a mass-flow meter. At a higher level of complexity, the density and Prandtl number of the fluid and the Reynolds number of the flow may be expected to vary widely, the velocity profile may be changeable because of the unavoidable presence of valves or pumps, or the fluid may sometimes exist in two phases; thereby, only a few designs of mass-flow meter may prove suitable.

CHAPTER 6. INTRODUCTION TO DYNAMIC PERFORMANCE. EXAMPLES OF SOME SIMPLE SYSTEMS

6.0 Introductory remarks. When the physical variable being measured by an instrument varies with time, one may desire either

- (i) a knowledge of the mean value of the variable over a given time interval, as deduced from the instrument indication, or
- (ii) a knowledge of the instantaneous value of the variable, as deduced from the simultaneous instrument indication.

The first case has been considered in Chapter 5. [Note N6.0] The second case requires more precise definitions of instantaneity and of simultaneity. Practically, these concepts imply averaging over a time interval Δt that is very short compared to the total time of observation. The choice of Δt is important because it affects the signal-to-noise ratio (Sec. 5.20) that can be achieved.

The second case will be illustrated in this chapter by some simple, basic systems; their behavior will then be treated in Chapter 7. The understanding thereby provided often serves to provide an understanding of the behavior of more complex variants or combinations of these systems, with little additional mathematics.

Simplicity of treatment of the systems is achieved if results are expressed in terms of the time t and of the dimensionless quantity "fraction of reference indication," which will be denoted by y ; it will then be found that only a few mathematical equations suffice to describe the dynamic performance of a variety of instruments, systems, or installations. The examples in this chapter illustrate this fact. The examples also provide an understanding of the instruments, of the factors that influence their dynamic behavior, and of the methods of analysis that can be applied to other situations.

In what follows, the first or second derivative of any variable with respect to time will be denoted by one or two dots, respectively, over that variable. Thus,

$$(6.0-1) \quad \dot{y} \equiv dy/dt; \quad \ddot{y} \equiv d^2y/dt^2$$

Generally, the equation describing the dynamic behavior of an instrument or system includes three terms or components. The first serves to describe the static behavior and the energy associated with the deviation of y from its equilibrium or static condition (the potential energy in mechanical systems). The second is indicative of the dissipation of energy and is usually associated with \dot{y} . The third is usually associated with \ddot{y} and, in mechanical systems, with the conservative kinetic energy.¹

¹In electrical systems, the two nondissipative forms of energy are those associated with the energies stored in pure capacitances and in pure inductances.

6.1 Linear spring-mass system. Seismometer, velocimeter, accelerometer, dynamometer, or pressure gauge.

In the system shown in Fig. 6.1, the elements are assumed to be

1. a linear spring having a spring constant k (force/distance), negligible mass, and negligible elastic errors;
2. a mass m ;
3. an oil film or equivalent device having negligible mass and producing a viscous force b per unit velocity in the x -direction;
4. an external force F that acts on mass m ;

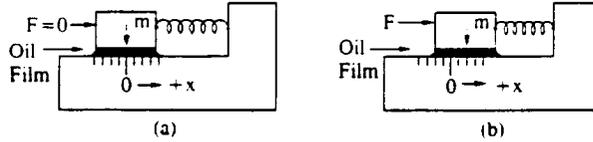


Figure 6.1—Linear spring-mass system.

5. an instrument base with scale that measures the horizontal deflection of mass m under force F .

Figure 6.1a shows the system at rest, with $F = 0$, and the scale adjusted so that pointer deflection $x = 0$. Figure 6.1b shows the system when $x > 0$ under the influence of the external force $F = F(t)$. We apply Newton's Second Law

$$(6.1-1a) \quad m\ddot{x} = \sum \text{forces.}$$

Then

$$\text{force exerted by oil film} = -b\dot{x}$$

$$\text{force exerted by spring} = -kx$$

$$\text{external force} = F$$

so that

$$(6.1-1b) \quad m\ddot{x} = F - b\dot{x} - kx .$$

Let x_m be the steady deflection under a steady force F_m (conveniently, these may be taken as full-scale values)

so that

$$(6.1-1c) \quad F_m = kx_m; \quad F = (F/F_m)kx_m$$

and Eq. (6.1-1b) may be rewritten as

$$(6.1-1d) \quad (m/k)\ddot{x} + (b/k)\dot{x} + x = (F/F_m)x_m .$$

Let

$$(6.1-2a) \quad x/x_m = y; \quad b/k = \tau; \quad m/k = 1/(4\pi^2 f_n^2) .$$

Then Eq. (6.1-1d) becomes

$$(6.1-2b) \quad \ddot{y}/(4\pi^2 f_n^2) + \tau\dot{y} + y = F/F_m .$$

This system serves as a model of several instruments:

A. When the instrument base oscillates in the x -direction at a frequency $f \gg f_n$, the model represents a *linear seismometer*. The magnitude of x is substantially the same as the magnitude of base displacement; force F is substantially zero. [Note N6.1]

B. When the instrument base oscillates in the x -direction at a frequency $f \approx f_n$, the model represents a *linear velocimeter*. The magnitude of the displacement x is approximately proportional to the velocity of the base.

C. When the instrument base oscillates in the x -direction at a frequency $f \ll f_n$, the model represents a *linear accelerometer*. A static calibration ($f = 0$) is possible, using the acceleration corresponding to the Earth's gravity, by turning the assembly $\pm 90^\circ$ from the horizontal position shown in Fig. 6.1. When the base is so turned, the model also represents a *spring balance*.

D. A *vibration meter* may use either a seismometer, velocimeter, or accelerometer as a pickup unit and then use electrical networks to produce single or double differentiation or integration to provide measurements of the other two variables.

E. If F is an external force, the model represents a *dynamometer* or *thrustmeter*. It represents a weighing device if it is turned 90° ; m then primarily represents the mass of the object being weighed.

F. If the mass m is in the form of a piston in a cylinder and the force F is the product of piston area and pressure on the piston face, the model represents a *pressure gauge*.

6.2 Correction for spring mass in Sec. 6.1. The expression for f_n in Eq. (6.1-2a) was based on the assumption that the spring had negligible mass. A more realistic expression would include the effect of the spring's mass m_s by adding some fraction of m_s to the quantity m in Eq. (6.1-2a). If m_s is uniformly distributed along the spring's length, as in a helical spring or a long bellows, then the fraction is 1/3. If the spring is a cantilever of constant cross section, loaded at the free end, the fraction is 33/140. If the spring is a circular diaphragm, and the supported mass occupies a small central area, the fraction is on the order of 1/4 if the outer edge is simply supported and is on the order of 1/6 if the outer edge is clamped. [Notes N6.2.1 and N6.2.2]

6.3 Rotational spring-mass system. Angular vibrometer, torquemeter, or galvanometer. The elements treated in Sec. 6.1 may be replaced by

1. a torsion spring having a spring constant K (torque per unit angle of twist) and with negligible mass and negligible elastic errors,
2. a mass with polar moment of inertia I ,
3. an oil film or equivalent device (like an eddy-current drag disc) having negligible mass and producing a resisting torque B per unit angular velocity,
4. an external torque T_e acting around the axis of the torsion spring and the center of rotation of the mass.

These are illustrated in Fig. 6.3.

The equation corresponding to Eq. (6.1-1b), that applies to the rotational system, is

$$(6.3-1) \quad I\ddot{\varphi} = T_e - B\dot{\varphi} - K\varphi$$

which can be put in the form

$$(6.3-2a) \quad \ddot{\varphi}/(4\pi^2 f_n^2) + \tau\dot{\varphi} + \varphi = T_e/T_{em}$$

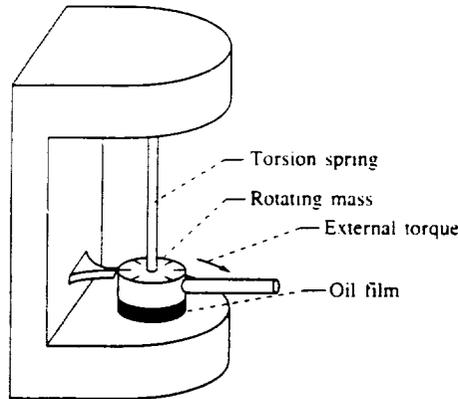


Figure 6.3—Rotational spring-mass system.

by the substitutions

$$(6.3-2b) \quad \varphi/\varphi_m = y; \quad B/K = \tau; \quad I/K = 1/(4\pi^2 f_n^2)$$

where φ_m is the steady angular deflection under a steady torque T_{em} . The quantity T_{em} may often conveniently be taken as the full-scale range of the device.

As in Sec. 6.2, if the torsion spring has uniformly distributed polar moment of inertia I_s , addition of $I_s/3$ to I in Eq. (6.3-2b) would yield a more nearly correct value of f_n .

As in Sec. 6.1, if the base undergoes a periodic oscillation at frequency f , the system is a model of a *vibration meter* whose mechanical deflection would be proportional to angular amplitude, angular velocity, or angular acceleration, depending on whether $f \gg f_n$, $f \approx f_n$, or $f \ll f_n$, respectively.

If the mass is subjected to a couple (e.g., if the mass is a motor armature or rotary-dynamometer armature), so that the twist of the spring is proportional to the magnitude of the couple, we have a model of a *torquemeter*.

The system is also a model of a *d'Arsonval galvanometer*, where B is the sum $B_1 + B_2$ of two factors, each of which arises from the facts that the motion of a conductor in a magnetic field generates voltages proportional to the velocity of relative motion and that these voltages generate currents whose reactions with the magnetic field produce forces that tend to oppose (damp) the motion. The factor B_1 is due to eddy currents generated in the moving-coil support and is not under the control of the user. The factor B_2 is due to the current in the coil, and is inversely proportional to the total resistance of the circuit of which the coil is a part; thus, B_2 is under the control of the user.

In the *d'Arsonval galvanometer*, a current i produces a torque proportional to i . If this torque is opposed by a mechanical moment FR produced by a force F acting at the end of an arm of radial length R , the arrangement is a model of an *electrobalance*, in which current i is adjusted until $\varphi = 0$. Then i is proportional to F .

6.4 U-tube manometer. In the manometer shown in Fig. 6.4, liquid of density ρ and viscosity η in a tube of inside radius r is subjected to a pressure difference P . The total length of the liquid column, from meniscus to meniscus, is l ; the curvature of the U-tube is assumed sufficiently small so that the effect of accelerations caused by curvilinear motion can be neglected; and Poiseuille's law is assumed to be applicable.² We apply Newton's Second Law to the liquid column when P is changing and one meniscus is deflected a distance x from the equilibrium position corresponding to $P = 0$.

$$\text{mass of liquid column} = \rho\pi r^2 l$$

$$\text{force exerted by gravity} = -2\rho g\pi r^2 x$$

²The assumption is valid only when l/r is very large and when the flow is always laminar.

force exerted by viscous drag = $-8\pi\eta l\dot{x}$

force exerted by external pressure = $\pi r^2 P$

Therefore,

$$(6.4-1a) \quad \rho\pi r^2 l \ddot{x} = -2\rho g\pi r^2 x - 8\pi\eta l \dot{x} + \pi r^2 P$$

If x_m is the steady deflection corresponding to a steady pressure difference P_m , so that

$$(6.4-1b) \quad 2\rho g x_m = P_m$$

then Eq. (6.4-1a) may be written in the form

$$(6.4-1c) \quad (l/g)\ddot{x} + [8\nu l/(gr^2)]\dot{x} + 2x = 2x_m(P/P_m)$$

where ν is the kinematic viscosity of the liquid, or

$$(6.4-2a) \quad \ddot{y}/(4\pi^2 f_n^2) + \tau\dot{y} + y = P/P_m$$

where

$$(6.4-2b) \quad y = x/x_m; \quad \tau = 4\nu l/(gr^2); \quad 1/(4\pi^2 f_n^2) = l/(2g)$$

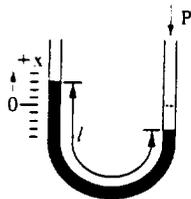


Figure 6.4.—U-tube manometer.

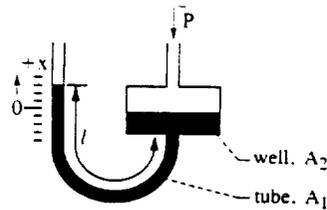


Figure 6.5.—Well-type manometer.

6.5 Well-type manometer. In the manometer shown in Fig. 6.5, the ratio A_1/A_2 between tube area and well area is made very small, so that

- (i) deflection x for a given pressure differential is almost twice what it would be in Fig. 6.4;
- (ii) the requirement in Fig. 6.4 that both legs of the U have exactly the same area or else that the displacements of both menisci be measured, is avoided.

Equations (6.4-1c) and (6.4-2b) still apply provided that l is replaced by

$$2(l + x)/(1 + A_1/A_2);$$

but the equation may be treated as linear (i.e., τ and f_n may be treated as constants) only when oscillations Δx about the average position are very much smaller than $(l + x)$.

The static calibration analogous to Eq. (6.4-1b) is

$$(6.5-1) \quad \rho g x_m (1 + A_1/A_2) = P_m \quad [\text{Footnote 3}]$$

³The scales of industrial well-type manometers are appropriately marked to correct for the area ratio and so directly to indicate the correct pressure.

- 6.6 Basic pressure transmission system.** In the system shown in Fig. 6.6, the components are assumed to be
1. a chamber containing a volume V of a compressible gas with specific-heat ratio γ , and initial pressure p_0 ;
 2. a long capillary tube, of cross-sectional area A , length l , filled with an incompressible fluid of density ρ , viscosity η , and kinematic viscosity ν ;
 3. a pressure $P = P(t)$ applied to the open end of the tube.

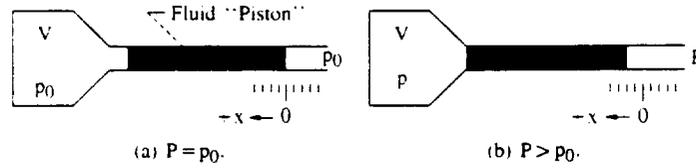


Figure 6.6—Pressure transmission system.

If we assume an adiabatic process within the chamber and Poiseuille's law within the tube, and if the position of the end of the fluid "piston" in the tube is assigned the coordinate $x = 0$ when external pressure is steady and equal to p_0 , the system becomes similar to the spring-mass system of Sec. 6.1, wherein the fluid in the tube becomes the mass m and the compressible gas in the chamber becomes a spring such that a piston displacement x (that produces a volume change Ax) also produces a pressure change $-(\gamma p_0/V)Ax$. We thus have, if an external pressure P produces a piston displacement x ,

$$\begin{aligned} \text{mass of piston} &= Al\rho \\ \text{viscous drag force on piston} &= -8\pi\eta l\dot{x} \\ \text{pressure in chamber} &= p_0 + (\gamma p_0/V)Ax \\ \text{external pressure} &= P = P(t) \end{aligned}$$

Application of Newton's Second Law leads to

$$(6.6-1a) \quad Al\rho\ddot{x} = [P - p_0 - (\gamma p_0/V)Ax]A - 8\pi\eta l\dot{x}$$

or

$$(6.6-1b) \quad l\rho\ddot{x} + 8\pi\eta l\dot{x}/A + (\gamma p_0/V)Ax = P - p_0$$

These equations may be converted into a description of the pressure p in the chamber by means of the adiabatic relation

$$(6.6-1c) \quad p = p_0 + Ax\gamma p_0/V$$

leading to

$$(6.6-1d) \quad [\rho lV/(A\gamma p_0)]\dot{p} + [8\pi\eta lV/(A^2\gamma p_0)]\dot{p} + p = P$$

If both sides of the equation are divided by a reference pressure p_m , there results

$$(6.6-2a) \quad \ddot{y}/(4\pi^2 f_n^2) + \tau\dot{y} + y = P/p_m$$

where

$$(6.6-2b) \quad 1/(4\pi^2 f_n^2) = \rho l V / (A \gamma p_0); \quad \tau = 8\pi\eta l V / (A^2 \gamma p_0); \quad y = p/p_m .$$

Equations (6.6-1) and (6.6-2) apply also when the tube contains the same gas as the chamber, if tube volume is negligible compared to chamber volume. However, it is then usually more convenient and more meaningful to express the quantities f_n and τ in terms of the speed of sound a and the kinematic viscosity ν :

$$(6.6-2c) \quad 1/(4\pi^2 f_n^2) = l V / (A a^2); \quad \tau = 8\pi\nu l V / (A^2 a^2)$$

These equations apply even to oval tubes.

As in Sec. 6.5, the quantities f_n and τ may be treated as constants only when the fluctuations in P are small compared to p_0 . [Note N6.6]

6.7 Modifications of the basic pressure transmission system. A number of elaborations of the basic system treated in Sec. 6.6 lead to several cases of practical interest. Combinations of these cases can also be treated, often by superposition. In the cases treated below, it will be assumed that the tube contains the same compressible gas as that in the chamber. The cross-sectional area of the tube will be denoted by A .

A. If the volume of the tube is small, but not negligible compared to the volume of the chamber, replace V in Eq. (6.6-2) with $V + Al/2$.

B. If the volume of the tube is comparable to or larger than the volume of the chamber, replace V in Eq. (6.6-2) with $V + 4Al/\pi^2$. This substitution extends even to the case when $V = 0$, in which case we have a *closed-end organ pipe*.

C. If l is on the order of magnitude of r , the effective radius (twice the hydraulic radius) of the tube, and if $Al \ll V$, replace l with $(l + \pi r/2)$ in the expression for f_n in Eq. (6.6-2) and assume $\tau = 0$. [Note N6.7] The system is called a *bottle resonator* or a *Helmholtz resonator with short neck*.

D. If $l \ll r$, replace l with $\pi r/2$ in the expression for f_n in Eq. (6.6-2) and assume $\tau = 0$. [Note N6.7] This system is called a *Helmholtz resonator*.

E. If the chamber wall is flexible, so that, at average pressure p_0 , a pressure increase Δp produces a chamber-volume increase $\alpha \Delta p$, then replace V with $(V + \alpha \gamma p_0)$ in Eq. (6.6-2). This model serves well when a pressure gauge is of the bellows or aneroid type.

6.8 Series electrical circuits. In the circuit of Fig. 6.8, containing inductance L , capacitance C , and resistance R in series with each other and with an externally applied voltage E , Kirchoff's law requires that the algebraic sum of all voltage drops around the loop be zero. Then, if charge q is circulating through the system to produce a current i ,

$$\text{voltage across } L = e_L = L(di/dt) = L\dot{q}$$

$$\text{voltage across } R = e_R = Ri = R\dot{q}$$

$$\text{voltage across } C = e_C = q/C$$

so that

$$(6.8-1a) \quad L\ddot{q} + R\dot{q} + q/C = E$$

or, since $q = Ce_C$,

$$(6.8-1b) \quad LC \ddot{e}_C + RC \dot{e}_C + e_C = E .$$

If we divide through by some constant reference voltage e_m , we may write

$$(6.8-2a) \quad \ddot{y}/(4\pi^2 f_n^2) + \tau\dot{y} + y = E/e_m$$

where

$$(6.8-2b) \quad 1/(4\pi^2 f_n^2) = LC; \quad \tau = RC; \quad y = e_C/e_m .$$

R-C Circuit

If inductance L is not present in the circuit, then Eqs. (6.8-2a) and (6.8-2b) become

$$(6.8-3a) \quad \tau\dot{y} + y = E/e_m$$

where

$$(6.8-3b) \quad \tau = RC; \quad y = e_C/e_m .$$

R-L Circuit

If capacitance C is not present in the circuit, then Eq. (6.8-1a) can be written as

$$(6.8-4) \quad (L/R)\dot{e}_R + e_R = E$$

and division of each term by some reference voltage e_m yields

$$(6.8-5a) \quad \tau\dot{y} + y = E/e_m$$

where

$$(6.8-5b) \quad \tau = L/R; \quad y = e_R/e_m .$$

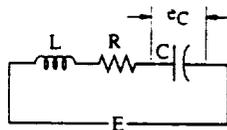


Figure 6.8.—Series electrical circuit.

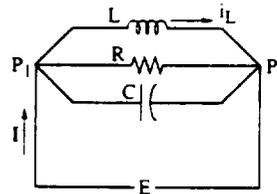


Figure 6.9.—Parallel electrical circuit.

6.9 Parallel electrical circuits. In the circuit of Fig. 6.9, wherein a current I passes through the parallel combination of inductance L , capacitance C , and resistance R , Kirchoff's law requires that the sum of all currents at point P_1 shall be zero. If the voltage between points P_1 and P_2 is E , then

$$(6.9-1a) \quad E = L(di_L/dt) = Ri_R = (1/C)q_C = (1/C) \int i_C dt$$

and Kirchoff's law

$$(6.9-1b) \quad i_C + i_R + i_L = I$$

can be written as

$$(6.9-1c) \quad LC(d^2i_L/dt^2) + (L/R)(di_L/dt) + i_L = I .$$

If we divide through by i_m , this may be written as

$$(6.9-2a) \quad \ddot{y}/(4\pi^2 f_n^2) + \tau \dot{y} + y = I/i_m$$

where

$$(6.9-2b) \quad 1/(4\pi^2 f_n^2) = LC; \quad \tau = L/R; \quad y = i_L/i_m .$$

R-L Circuit

If capacitance C is not present in the circuit, then Eqs. (6.9-2a) and (6.9-2b) can be written as

$$(6.9-3a) \quad \tau \dot{y} + y = I/i_m$$

where

$$(6.9-3b) \quad \tau = L/R; \quad y = i_L/i_m .$$

R-C Circuit

If inductance L is not present in the circuit, then Eq. (6.9-1b) can be written as

$$(6.9-4) \quad RC(di_R/dt) + i_R = I$$

and division of each term by some reference current i_m leads to

$$(6.9-5a) \quad \tau \dot{y} + y = I/i_m$$

where

$$(6.9-5b) \quad \tau = RC; \quad y = i_R/i_m .$$

6.10 Mass moving through a viscous medium. If a force F causes a mass m to move through a viscous medium that exerts a resisting force bv proportional to the velocity v of motion, application of Newton's law yields

$$(6.10-1a) \quad m\dot{v} = F - bv .$$

Under a steady force F_m , the mass would ultimately acquire a constant velocity v_m given by

$$(6.10-1b) \quad F_m = bv_m .$$

Consequently, Eq. (6.10-1a) may be written as

$$(6.10-2a) \quad \tau \dot{y} + y = F/F_m$$

where

$$(6.10-2b) \quad \tau = m/b; \quad y = v/v_m .$$

These relations are applicable to the phenomenon of sedimentation under the action of gravity and to the falling-ball viscometer.

6.11 Filling of a gas container. In Fig. 6.11, a chamber of volume V , initially at pressure p_0 , is filled through a restriction connecting the chamber to a source of higher pressure P . The following conditions are assumed to exist:

1. The fluid is a perfect gas.
2. Gas motion is so slow that inertia effects can be neglected.
3. The exchange process is isothermal, at temperature θ .
4. The restriction through which the filling takes place is a linear one, in which mass flow rate \dot{m} through the restriction is proportional to pressure difference $P - p$, where P is upstream pressure and p is the pressure in the chamber:

$$(6.11-1) \quad \dot{m} = C \cdot (P - p)$$

where C has the dimensions of (mass flow rate)/(pressure difference). [Note N6.11.1]

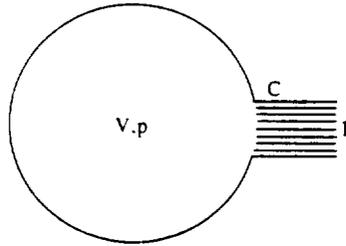


Figure 6.11.—Filling of a gas container.

Since the rate of accumulation of mass in the chamber is

$$(6.11-2) \quad \dot{m} = V \cdot \dot{\rho} = V \dot{p} M / (R_0 \theta) ,$$

where M is relative molar mass and R_0 is the universal gas constant, the law of conservation of mass requires that

$$(6.11-3a) \quad V \dot{p} M / (R_0 \theta) = C \cdot (P - p); \quad C \equiv d\dot{m}/dp$$

so that

$$(6.11-3b) \quad [VM / (CR_0\theta)] \dot{p} + p = P .$$

Dividing through by some reference pressure p_m leads to

$$(6.11-4a) \quad \tau \dot{y} + y = P/p_m$$

where

$$(6.11-4b) \quad \tau = VM / (CR_0\theta); \quad y = p/p_m .$$

These results are applicable to the calibration of standard leaks and flow impedances. [Note N6.11.2]

Example 6.11a. If the restriction is a capillary tube of cross-sectional area A , and if $p_0 > 1$ mbar,

$$C = A^2 / (8\pi\nu)$$

where the kinematic viscosity ν must be chosen as the value at a particular location. [Note N6.11.3]

Then

$$\tau = 8\pi\nu lVM / (A^2 R_0 \theta) \quad .$$

Example 6.11b. If the restriction is a knife-edged orifice of area A , and if $p_0 < 1$ μ bar,

$$C \approx 2A / (\pi c_A)$$

where c_A is the arithmetic mean speed of the molecules and is given by

$$c_A^2 = 8R_0\theta / (\pi M) \quad .$$

Then

$$\tau = 4V / (Ac_A) \quad .$$

(c_A is 468.0 m/s if $M = 29.0$ and $\theta = 300$ K.)

6.12 Thermometer bulb. Assume that a thermometer bulb of mass \bar{m} , specific thermal capacity c (energy to produce unit temperature rise in unit mass), very high thermal conductivity, and exposed surface area A is at temperature θ and receives heat through its exposed surface by conduction along a column of matter of cross-sectional area A , length l , and thermal conductivity k , whose far end is maintained at temperature Θ (Fig. 6.12a). At the interface between the column and the bulb, the rate of heat conduction is equal to the rate at which heat is stored in the bulb.

Rate of heat conduction through the column = $kA(\Theta - \theta)/l$

Rate of storage of heat in bulb = $mc \cdot d\theta/dt$

Consequently,

$$(6.12-1) \quad mc\dot{\theta} = kA(\Theta - \theta)/l \quad .$$

If each term is divided by some reference temperature θ_m , the relation may be expressed as

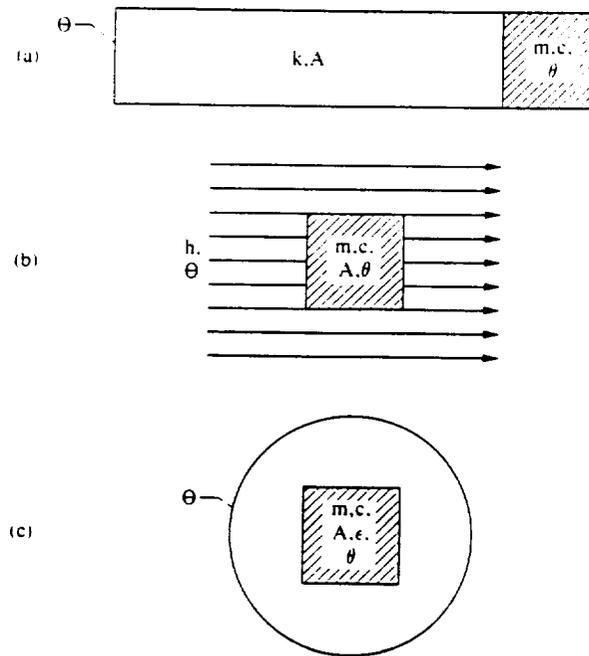
$$(6.12-2a) \quad \tau\dot{y} + y = \Theta/\theta_m$$

where

$$(6.12-2b) \quad \tau = mcl / (kA); \quad y = \theta/\theta_m \quad .$$

If the means of transferring heat to the bulb is by gas convection (Fig. 6.12b) and the convective heat transfer coefficient is h (power per unit area per unit temperature difference), then the equality of heat transfer rate and heat storage rate is expressed by

$$(6.12-3) \quad mc\dot{\theta} = hA(\Theta - \theta) \quad .$$



(a) Conductive heat transfer.
 (b) Convective heat transfer.
 (c) Radiative heat transfer.

Figure 6.12.—Thermometer bulb.

Division of each term by some reference temperature θ_m leads to

$$(6.12-4a) \quad \tau \dot{y} + y = \Theta/\theta_m$$

where

$$(6.12-4b) \quad \tau = mc/(hA); \quad y = \theta/\theta_m .$$

If the surface of the bulb has an emittance ϵ and if the bulb is surrounded by an enclosure at temperature Θ (Fig. 6.12c), the rate of energy exchange by radiation is such that

$$(6.12-5) \quad mc\dot{\theta} = \sigma\epsilon A(\Theta^4 - \theta^4)$$

where σ is the Stefan-Boltzmann constant. If $(\Theta - \theta)/\Theta \ll 1$, an approximation is

$$(6.12-6) \quad mc\dot{\theta} = 4\sigma\epsilon A\Theta^3(\Theta - \theta) .$$

Division of each term by some reference temperature θ_m leads to

$$(6.12-7a) \quad \tau \dot{y} + y = \Theta/\theta_m$$

where

$$(6.12-7b) \quad \tau = mc/(4\sigma\epsilon A\Theta^3); \quad y = \theta/\theta_m .$$

6.13 Summary. All of the examples of simple dynamical systems in this chapter have led to the expressions of the form

$$(6.13-1) \quad \ddot{y}/(4\pi^2f_n^2) + \tau\dot{y} + y = Y$$

or of the form

$$(6.13-2) \quad \tau\dot{y} + y = Y$$

where y is a dimensionless quantity representing instrument indication and Y is a dimensionless quantity representing the physical variable being measured. The parameter f_n will be termed the “natural frequency”; the parameter τ will be termed the “time constant”. These parameters are significant only when Y is varying with time; if it is not, then we have the static conditions considered in Chapter 5.

The values of f_n and τ for some of the particular systems treated here are summarized in Table 6.13.

TABLE 6.13.—PARAMETERS OF SOME DYNAMICAL SYSTEMS

System	Variable	τ	$1/(4\pi^2f_n^2)$
Linear spring-mass	x	b/k	m/k
Torsional spring-mass	ϕ	B/K	I/K
Manometer	x	$4vl/(gr^2)$	$l/(2g)$
Pressure transmission	P	$8\pi\nu lV/(A^2a^2)$	$lV/(Aa^2)$
Series L, R, C	e_C	RC	LC
Series R, C	e_C	RC	
Series L, R	e_R	L/R	
Parallel L, R, C	i_L	L/R	LC
Parallel L, R	i_L	L/R	
Parallel R, C	i_R	RC	
Mass in viscous medium	v	m/b	
Filling a gas container, continuum	p	$8\pi\nu lV/(A^2R_0\theta/M)$	
Filling a gas container, free-molecule	p	$4V/(Ac_A)$	
Thermometer bulb, conduction	θ	$mc l/(kA)$	
Thermometer bulb, convection	θ	$mc/(hA)$	
Thermometer bulb, radiation	θ	$mc/(4\sigma\epsilon A\theta^3)$	

NOTES FOR CHAPTER 6

N6.0. Considerations of linearity are important. Thus, a correct arithmetic mean will be obtained if the instrument indication varies linearly with the variable being measured; if it does not, then the desired accuracy can be achieved only if the variations of the variable are of sufficiently small magnitude.

N6.1. “Seismographs” used to record earth motions that occur in earthquakes may actually be velocimeters or accelerometers because the amount of earth displacement may be too large to be recordable by any practical seismometer.

N6.2.1. More precisely, if a = outer edge diameter and b = central area diameter, the fraction is given in the following table

$a/b =$	5	10	20
Supported edge	0.31	0.27	0.25
Clamped edge	0.20	0.16	0.14

N6.2.2. When a flush-diaphragm-type pressure gauge is immersed in a liquid, the mass of the liquid next to the diaphragm becomes an additional part of the distributed mass m_r . An approximate estimate of the effect of this associated liquid mass can be made by assuming that a hemisphere of liquid, erected on the diaphragm, moves with the diaphragm.

N6.6. It is sometimes helpful to replace Eq. (6.6-1) with an equation for the volumetric displacement

$$u = Ax$$

or an equation for the mass displacement

$$M = \rho Ax .$$

The respective equations then are

$$(N6.6-1) \quad (l\rho/A)\ddot{u} + (8\pi\eta l/A^2)\dot{u} + (\gamma p_0/V)u = P - p_0$$

and

$$(N6.6-2) \quad (l/A)\ddot{M} + (8\pi\nu l/A^2)\dot{M} + (a^2/V)M = P - p_0 .$$

Equation (6.6-2) remains unchanged.

N6.7. The effect of energy dissipation because of fluid viscosity is usually negligible. If the resonator opens into a medium of substantially infinite extent, any acoustic oscillations that exist after removal of the initiating source of energy will die out principally because of the radiation of energy into this infinite medium. The effective value of τ is given by

$$\tau \approx A/(4\pi a l)$$

for a resonator with short neck, and by

$$\tau^2 \approx A/(4\pi^3 a^2)$$

for a resonator with neck of negligible length.

N6.11.1. The value of C depends on the geometric form of the restriction and on the intrinsic and extrinsic properties of the gas.

Typical intrinsic properties are viscosity, relative molar mass (molecular weight), and Prandtl number.

Typical extrinsic properties are density, temperature, pressure, speed of sound, and Reynolds number.

N6.11.2. Thereby, it is not necessary to measure mass flow rate in order to determine the flow resistance C . It is necessary only that V and t be accurate, that P be constant, and that the pressure gauge be linear. In some practical applications, it may be necessary to correct for any rate of pressure rise in V due to outgassing from the inner surface of the chamber.

N6.11.3. The location preferred is that at which ν is likely to change least during the course of the filling operation. Usually, this location is the entrance end of the capillary. Other locations are (a) the exit, or (b) a point where the pressure is the mean of the entrance and exit pressures.

Appreciable changes in kinematic viscosity during the course of the filling operation make Eq. (6.11-4) nonlinear; such nonlinearity is discussed in Sec. 10.3.

CHAPTER 7. DYNAMIC PERFORMANCE OF BASIC SYSTEMS

7.0 Introduction and terminology. A study of the behavior of systems described by Eqs. (6.13-1) and (6.13-2) is useful because

- (a) the behavior is completely characterized by no more than two independent parameters and hence can be easily understood and remembered;
- (b) more complex systems can often be represented as combinations of the systems described by Eqs. (6.13-1) and (6.13-2) so that an intuitive understanding of the behavior of the more complex system can be derived from an understanding of the behavior of the simpler component systems;
- (c) an understanding of dynamic behavior helps the attainment of several possible goals:
 - (i) adequately accurate determination of the average value of a quantity that is changing with time,
 - (ii) adequately accurate determination of the instantaneous value of such a quantity,
 - (iii) adequate suppression of random or systematic fluctuations (usually categorized as “noise” or “hum”) that tend to prevent such accurate instantaneous measurements.

In practical situations where dynamic measurements are to be made, the balance between “signal” and “noise” is always necessary, and requires a human judgment that also considers convenience, cost, and reliability.

The following symbols will be used frequently in this chapter:

f	frequency of $Y(t)$
f_c	corner frequency
f_n	natural frequency
f_r	resonant frequency
I	impulse
t	time
T	period of $Y(t)$
T_n	natural period
T_r	resonant period
y	dimensionless variable proportional to the instrument indication (instrument response)
Y	dimensionless variable proportional to the physical quantity being measured. $Y(t)$ is also called the forcing function.
α	$\sqrt{ 1 - \zeta^2 }$
ϵ	base of Napierian logarithms
Λ	radius of curvature
τ	time constant
ζ	fraction of critical damping
ω	angular frequency of $Y(t)$
ω_n	natural angular frequency
ω_r	resonant angular frequency

Other symbols used occasionally will be defined as they are introduced. As in Chapter 6, one or two dots over a variable denotes the first or second derivative with respect to time.

7.1 Basic equations. The basic first-order system will be defined as one obeying the equation

$$(7.1-1) \quad \tau \dot{y} + y = Y \equiv Y(t)$$

where τ is a constant.

The basic second-order system will be defined as one obeying the equation

$$(7.1-2a) \quad \ddot{y}/\omega_n^2 + \tau \dot{y} + y = Y \equiv Y(t) \quad .$$

This equation can be written in several other forms, three of which are shown here

$$(7.1-2b) \quad \ddot{y}/(4\pi^2 f_n^2) + \tau \dot{y} + y = Y$$

$$(7.1-2c) \quad \ddot{y}/\omega_n^2 + (2\zeta/\omega_n)\dot{y} + y = Y$$

$$(7.1-2d) \quad \ddot{y}T_n^2/(4\pi^2) + (\zeta T_n/\pi)\dot{y} + y = Y$$

The various constants are related by

$$(7.1-3) \quad \omega_n = 2\pi f_n = 2\pi/T_n; \quad \tau = 2\zeta/\omega_n = \zeta T_n/\pi \quad .$$

The dynamic behavior problem is treated in two steps:

1. Given $Y(t)$, one finds $y(t)$. The difference $(y-Y)$ represents the instantaneous error of measurement.
2. One then determines what values of the parameters in (7.1-3) will render the error acceptably small.

7.2 Parameters affecting response. The response of a first-order system to any $Y(t)$ is determined entirely by the value of τ . The response of a second-order system to any $Y(t)$ is determined entirely by two parameters: one of these is chosen from the group (f_n, T_n, ω_n) ; the other is either τ or ζ . When $\zeta \leq 1$, it is usually more convenient to use ζ in mathematical descriptions of the response and to express time in terms of T_n . When $\zeta > 1$, it is usually more convenient to use τ ; in fact, when $\zeta \gg 1$, it is often adequate to treat the system as a first-order system described by Eq. (7.1-1).

7.3 Influence of prior values of y and its derivatives. Assume that $Y(t)$ is known for $t > t_0$. Let $y_0 \equiv y(t_0)$, $\dot{y}_0 \equiv \dot{y}(t_0)$, $\ddot{y}_0 \equiv \ddot{y}(t_0)$. If the system is a first-order one, $y(t)$ for $t \geq t_0$ will depend only on y_0 . If the system is a second-order one, $y(t)$ for $t \geq t_0$ will depend only on y_0 and \dot{y}_0 . If the system is an n^{th} -order one, $y(t)$ for $t \geq t_0$ will depend on y_0 and on the first $(n-1)$ derivatives of y at time t_0 .

An alternative phrasing is to say that $y(t)$ for a first-order system is capable of executing abrupt changes in direction (i.e., y must be continuous, but \dot{y} need not be continuous); in a second-order system, abrupt

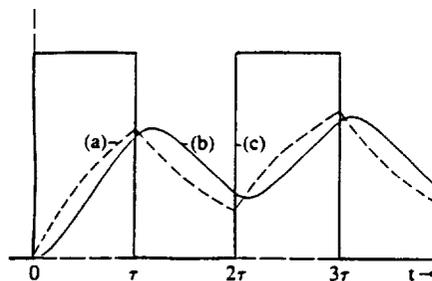


Figure 7.3.—Response to a square wave of two systems with the same time constant, τ . (a) first-order system; (b) critically-damped second-order system; (c) forcing function with period 2τ .

changes in direction cannot occur, although abrupt changes in the second derivative may occur (i.e., y and \dot{y} must be continuous, but \ddot{y} need not be continuous). This behavior is illustrated in Fig. 7.3, which shows the response y of first-order and second-order systems to abrupt changes in Y . Sharp corners in $y(t)$ are possible only for a first-order system; they are not possible in systems of higher order. In an n^{th} -order system, abrupt changes (sharp corners) are not possible in the first $(n-1)$ derivatives.

7.4 Higher-order systems. If the dynamic behavior of a physical system is described by a linear differential equation of the n^{th} order, that description may be rewritten as a combination of first-order and second-order equations representing corresponding first-order and second-order subsystems. In one such combination, the subsystems are in tandem, so that the indication (output) of one subsystem becomes the input of the next subsystem; the output of the last subsystem is the desired response y of the entire system to the initial input Y . Such a combination may be termed a “serial combination”.

In another combination, appropriate modifications of the input Y are delivered to each subsystem and the outputs of all subsystems are summed to yield the indication y . Such a combination may be termed a “parallel combination”. [Note N7.4]

7.5 Transient and steady-state responses. For any physical system, the response $y(t)$ to an applied $Y(t)$ is often conveniently treated as composed of two parts: a *transient response* whose magnitude and character depend on the conditions (termed the “initial conditions”) existing at time $t = t_0$ when Y was first applied; and a *steady-state response* whose magnitude and character are not influenced by the initial conditions and depend almost entirely on the current behavior of $Y(t)$. The terms “transient” and “steady state” are used because, in most cases, the transient response is of greatest magnitude during the time immediately after t_0 and ultimately dies out (i.e., becomes negligibly small after a sufficiently long time has elapsed), whereas the steady state remains after the transient has died out and persists as long as Y persists. In a strict mathematical sense, the steady-state response is an asymptotic condition approached as $t \rightarrow \infty$. Practically, it often becomes indistinguishable from the asymptotic value after a time equal to a few multiples of τ .

The applied quantity Y , which is also termed the “forcing function”, represents the true value of a quantity to be measured. The response y represents the indicated value of that quantity. In any relation of the form

$$(7.5-1a) \quad y = Y - \Delta \quad ,$$

Δ represents the correction to be added algebraically to the indication y in order to find the true value Y .

An elaboration of Eq. (7.5-1a) is

$$(7.5-1b) \quad y = Y - \Delta_{ss} - \Delta_T$$

where Δ_{ss} and Δ_T represent the steady-state and the transient corrections, respectively. Their distinctive features are summarized here.

Δ_{ss} :

- (1) Mathematically, it is an asymptote approached as $t \rightarrow \infty$.
- (2) Practically, it may be approached to within acceptably small limits of deviation in a time determined by the values of the parameters listed in Sec. 7.1.
- (3) It does not depend on the conditions existing at the initiation of the measurement.
- (4) In most practical cases, it is of sufficiently simple form so that the application of corrections is convenient.

Δ_T :

- (1) Mathematically, $\Delta_T \rightarrow 0$ as $t \rightarrow \infty$.
- (2) Practically, it reaches any specified small magnitude in a time determined by the values of the parameters listed in Sec. 7.1.
- (3) It depends on the conditions existing at the initiation of the measurement.
- (4) When it is of sufficiently simple form, it may be convenient to apply it as a correction to Y in order to deduce y ; at other times, it may be more convenient to wait until it becomes negligible in comparison with $y + \Delta_{ss}$ before the deduction of Y is attempted.

If the steady-state response is known for each of several forcing functions, like those treated in the sections that follow, the steady-state response to the sum of the individual forcing functions is merely the sum of the individual steady-state responses.

The transient responses may not be treated in this additive manner.

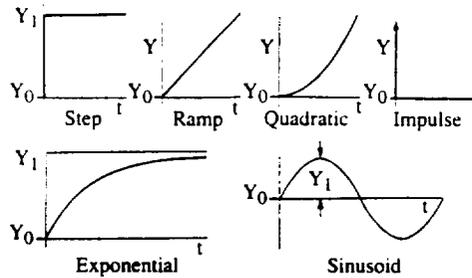


Figure 7.6.—Some forms of the forcing function.

7.6 Forms of the forcing function. In the remainder of this chapter, both transient and steady-state responses will be indicated for several forms of the forcing function Y . These are illustrated in Fig. 7.6. For ease and clarity of presentation, the initial time t_0 will always be taken as $t = 0$.

- (a) *Step.* A sudden change in Y from an initial constant value Y_0 to another constant value Y_1 .
- (b) *Exponential.* An approach of Y from an initial constant value Y_0 to an asymptotic value Y_1 in an exponential manner.
- (c) *Ramp.* A linear change of Y with time, starting from an initial constant value Y_0 .
- (d) *Quadratic parabola.* A parabolic change of Y with time, starting from an initial constant value of Y_0 .
- (e) *Impulse.* A momentary step change in Y , from an initial value Y_0 to a final value Y_1 , that is of such short duration Δt that $\Delta t \approx 0$, $Y_1 \approx \infty$, but $(Y_1 - Y_0) \cdot \Delta t = I$ is finite. The product I is termed the “impulse”.
- (f) *Sinusoid.* A sinusoidal variation in Y with amplitude Y_1 and average value Y_0 . This form is particularly useful because any periodic, continuous physical phenomenon can be represented by a summation of sinusoidal signals of appropriate phase and amplitude.

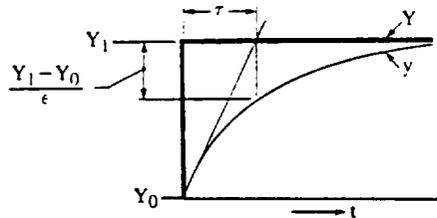


Figure 7.7.—Response of a first-order system to a step change.

7.7 Response of a first-order system to a step change. (Fig. 7.7) If

$$(7.7-1) \quad Y = Y_1 \quad \text{for } t > 0$$

$$\text{and } Y = y = Y_0 \quad \text{at } t = 0$$

then

$$(7.7-2) \quad y = Y_1 - (Y_1 - Y_0)e^{-t/\tau}$$

is the complete solution.

The asymptote is Y_1 .

The initial slope $\dot{y}(0)$ is $(Y_1 - Y_0)/\tau$.

If the tangent dy/dt is drawn at $t = 0$, it will intersect the asymptote Y_1 at $t = \tau$.

In time τ , $(y - Y_0)$ will be $(1 - 1/e)(Y_1 - Y_0)$; in other words, 63.2 percent of the step will have been completed. In time 2τ , 3τ , 4τ , 5τ , approximately 86, 95, 98, 99.3 percent, respectively, of the step $(Y_1 - Y_0)$ will have been completed. [Note N7.7]

7.8 Response of a first-order system to an exponential change. (Fig. 7.8) If

$$(7.8-1) \quad Y = Y_1 - (Y_1 - Y_0)e^{-t/\lambda} \quad \text{for } t > 0$$

$$\text{and} \quad Y = y = Y_0 \quad \text{at } t = 0$$

then

$$(7.8-2a) \quad y = Y_1 - (Y_1 - Y_0)(\lambda e^{-t/\lambda} - \tau e^{-t/\tau})/(\lambda - \tau) \quad \text{if } \lambda \neq \tau$$

$$(7.8-2b) \quad y = Y_1 - (Y_1 - Y_0)(1 + t/\tau)e^{-t/\tau} \quad \text{if } \lambda = \tau$$

The response y starts out at zero slope and passes through a point of inflection at (t_2, y_2) . After this point, y approaches Y_1 asymptotically in a manner resembling an exponential curve with time constant on the order of $(\tau + \lambda)$. [Note N7.8]

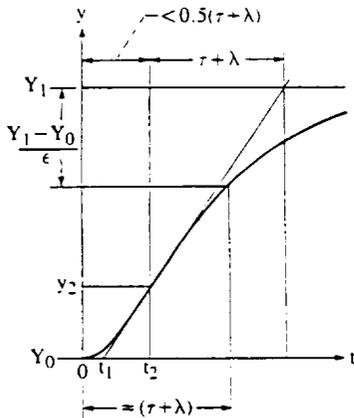


Figure 7.8.—Response of a first-order system to an exponential forcing function.

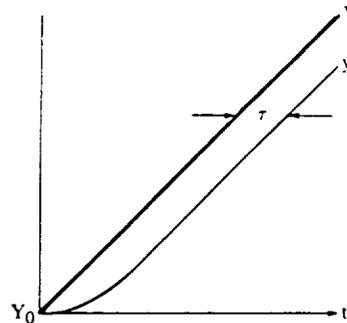


Figure 7.9.—Response of a first-order system to a ramp.

7.9 Response of a first-order system to a ramp. (Fig. 7.9) If

$$(7.9-1) \quad Y = Y_0 + at \quad \text{for } t > 0$$

$$\text{and} \quad Y = y = Y_0 \quad \text{at } t = 0$$

then

$$(7.9-2) \quad y = Y_0 + a(t - \tau) + a\tau e^{-t/\tau}$$

The steady state is a ramp parallel to Y but lagging behind it by the constant lag time τ . The term containing the exponential is the transient response; the response y starts out with zero slope at $t = 0$ and approaches the steady-state condition.

At times $\tau, 2\tau, 3\tau, 4\tau, 5\tau$, the respective lag time between y and Y is 63, 86, 95, 98, 99.3 percent of τ . (These percentages are the same as those quoted in Sec. 7.7). The differences between the ordinates at these same times is $a\tau$ times the respective percentages. [Note N7.9]

7.10 Response of a first-order system to a quadratic forcing function. (Fig. 7.10) If

$$(7.10-1) \quad Y = Y_0 + bt^2 \quad \text{for } t > 0$$

$$\text{and} \quad y = y = Y_0 \quad \text{at } t = 0,$$

the forcing function is a quadratic parabola with vertex at $(0, Y_0)$. The response is given by

$$(7.10-2) \quad y = Y_0 + b\tau^2 + b(t - \tau)^2 - 2b\tau^2 e^{-t/\tau}.$$

The steady state is a quadratic parabola of the same shape as the forcing function but with the vertex shifted to $(\tau, Y_0 + b\tau^2)$. This vertex lies on the original forcing function. The lag between the response and the forcing function approaches τ , regardless of the initial value of y .

The term containing the exponential represents the transient response. At times $t = \tau, 2\tau, 3\tau, 4\tau, 5\tau$, the respective lag between y and Y is $0.49\tau, 0.69\tau, 0.79\tau, 0.84\tau$, and 0.89τ .

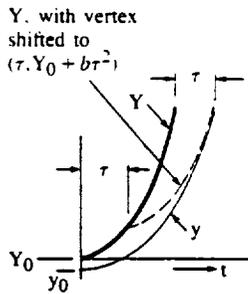


Figure 7.10.—Response of a first-order system to a quadratic forcing function.

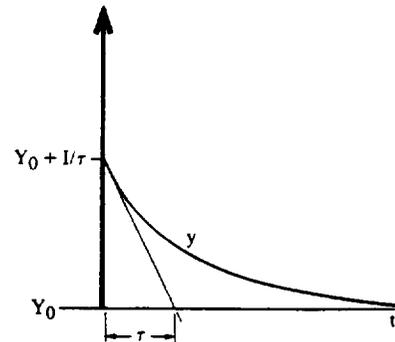


Figure 7.11.—Response of a first-order system to an impulse.

7.11 Response of a first-order system to an impulse. (Fig. 7-11) An impulse is defined as a change in Y that is of very high amplitude and very short duration (very much less than τ), but such that the area under $Y(t)$ is a finite constant I .

A rectangular pulse of very short duration Δt may be described by the equations

$$(7.11-1a) \quad Y - Y_0 = I/\Delta t \quad \text{for } 0 \leq t \leq \Delta t \ll \tau$$

$$(7.11-1b) \quad y = Y = Y_0 \quad \text{at } t = 0$$

$$Y = Y_0 \quad \text{for } t > \Delta t$$

The response y will reach a maximum value of $Y_0 + I/\tau$ at the end of the interval Δt ; thereafter, it will decay exponentially to an asymptote Y_0 , behaving as though it were responding to a step change from an initial value of $Y_0 + I/\tau$ to a final value of Y_0 , in accordance with Sec. 7.7. [Note N7.11]

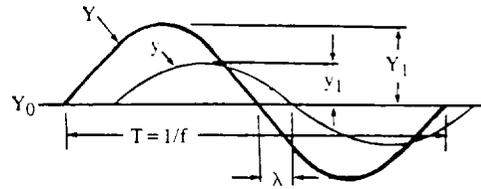


Figure 7.12.1.—Steady-state response of a first-order system to a sinusoid.

7.12 Response of a first-order system to a sinusoid. (Fig. 7.12.1) For this forcing function, only the steady-state response will be considered. The transient response is only of academic interest because it depends on the exact phase of the sinusoid at the instant that it is imposed on the system [Note N7.12.1]. If the forcing function Y is described by the equation

$$(7.12-1) \quad Y - Y_0 = Y_1 \sin(2\pi ft) = Y_1 \sin(\omega t),$$

the steady-state response will be given by

$$(7.12-2a) \quad y - Y_0 = y_1 \sin(\omega t - \phi) \\ = y_1 \sin[\omega(t - \lambda)]$$

where y_1/Y_1 is termed the amplitude ratio

ϕ is termed the phase angle

λ is termed the time lag

These quantities are given by

$$(7.12-2b) \quad y_1/Y_1 = (1 + \omega^2\tau^2)^{-1/2}$$

$$(7.12-2c) \quad \phi = \tan^{-1}(\omega\tau) = \cos^{-1}(1 + \omega^2\tau^2)^{-1/2}$$

$$(7.12-2d) \quad \lambda = (1/\omega) \tan^{-1}(\omega\tau)$$

The dependence of these three quantities on the value of $\omega\tau$ is shown in Fig. 7.12.2.

An alternative way of graphing the steady-state response is to make a polar plot of amplitude ratio *versus* phase angle. Such a graph is termed a *Nyquist diagram*. For the first-order system, the diagram is shown in Fig. 7.12.3. It is a semicircle based on the radius vector $y_1/Y_1 = 1$ at angle $\phi = 0$, and lies entirely in the first negative quadrant.

From Eq. (7.12-2) and Figs. 7.12.2 and 7.12.3, some useful conclusions, that depend on the value of $\omega\tau$, are indicated here:

- (1) If $\omega\tau \ll 1$, the amplitude ratio is reduced only slightly (often negligibly) and the time lag is approximately equal to τ .
- (2) If $\omega\tau = 1$, the amplitude ratio is $1/\sqrt{2}$, the phase angle is 45° , and the time lag is $\pi\tau/4$. The amplitude ratio and the phase angle are independent of τ .
- (3) If $\omega\tau \gg 1$, the amplitude ratio is approximately $1/(\omega\tau)$, the phase angle is almost 90° , and the time lag is approximately $1/(4f)$. On the logarithmic graph of Fig. 7.12.2a, the asymptote of the graph is a line of slope -1 ; this asymptote intersects the ordinate $y_1/Y_1 = 1$ at $\omega\tau = 1$. The frequency f_c at which this intersection occurs, given by

$$(7.12-3) \quad 2\pi f_c \tau = 1,$$

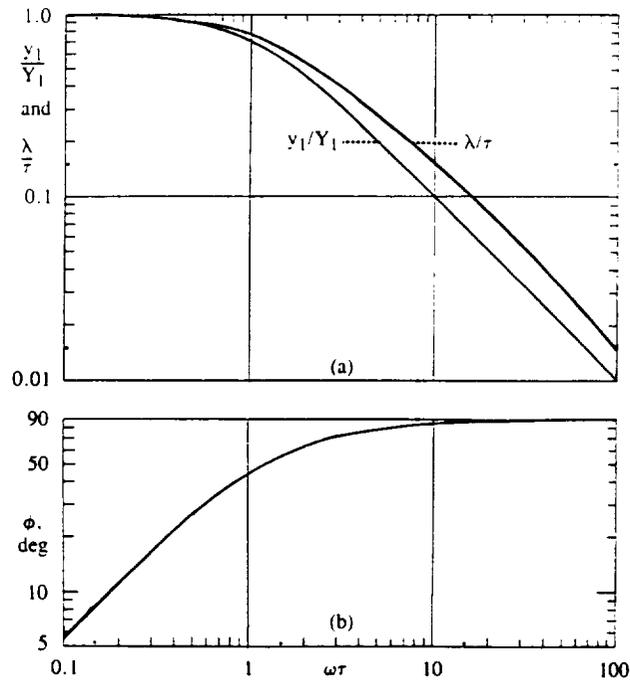


Figure 7.12.2.—Amplitude ratio y_1/Y_1 , time lag λ , and phase angle ϕ of a first-order system's response to a sinusoid.

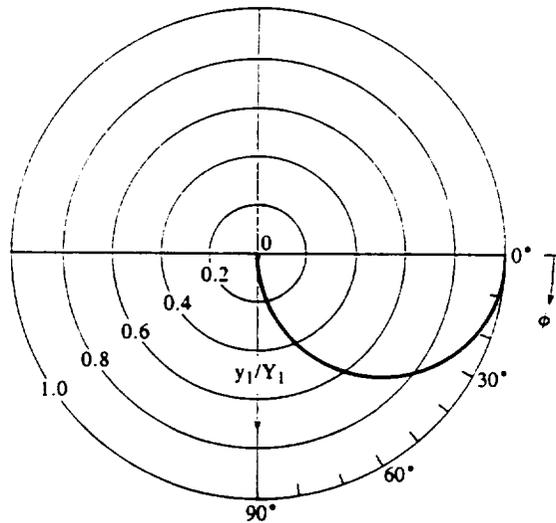


Figure 7.12.3.—Amplitude ratio y_1/Y_1 versus phase angle ϕ of response to a sinusoid by a first-order system.

is the *corner frequency*. The asymptote with slope -1 is said to have “a slope of 6 dB per octave” because, along this straight line, doubling the frequency would produce a 6-dB reduction in $(y_1/Y_1)^2$, which, in several of the physical situations treated in Chapter 6, would be proportional to power.

In simplified descriptions of the response of a first-order system to a sinusoid, the graph of Fig. 7.12.2a is replaced by two straight lines: a line of slope -1 passing through the point (1,1) for $\omega\tau \geq 1$, and a horizontal line through the same point for $\omega\tau \leq 1$.

In an electrical circuit involving capacitance C and resistance R , the quantity

$$(7.12-4) \quad D = \omega\tau = \omega RC$$

may be a distinctive parameter of the circuit. In particular, if R_s is the effective series resistance of an imperfect capacitor of capacitance C , the quantity $\omega R_s C$ is the *dissipation factor* of the capacitor. [Note N7.12.2]

In an electrical circuit involving inductance L and resistance R , the quantity

$$(7.12-5) \quad Q = \omega\tau = \omega L/R$$

may be a distinctive parameter of the circuit. In particular, if R_s is the effective resistance of an imperfect inductor of inductance L , the quantity $\omega L/R_s$ is the *storage factor* of the inductor. [Note N7.12.3] The *dissipation factor* of an inductor is the reciprocal of the storage factor.

7.13 Response of a first-order system to any forcing function. If a forcing function $Y(t)$ can be represented, over some time interval, by a sum of sinusoids

$$(7.13-1) \quad Y = \Sigma Y_j \sin(\omega_j t - \psi_j)$$

and if only terms for which $\omega_j \tau < 1$ have significantly large amplitude Y_j , then the record $y(t)$ yielded by an instrument with time constant τ can be used to approximate $Y(t)$ by the formula

$$(7.13-2) \quad Y(t) \approx y(t + \tau) \quad .$$

This situation is common in instrumentation because an instrument is usually selected, *a priori*, to provide good fidelity in measuring Y . Equation (7.13-2) then provides a first-order correction to improve accuracy.

In practice, the validity of the procedure is established as follows:

- (1) The record of $y(t)$ is inspected to identify oscillation frequencies that may be apparent.
- (2) It is verified that the corresponding apparent amplitudes of these oscillations are significantly large only for frequencies for which $\omega_j \tau < 1$.
- (3) It is further verified that $y(t)$ shows no sharp corners, representing abrupt changes in direction, that would imply the existence of transients of appreciable magnitude.

The following alternative procedure is useful when no oscillation frequencies are readily identifiable, but there remains the question of whether $Y(t)$ is deducible from $y(t)$:

Given the Cartesian graph of $y(t)$ produced by a system of known time constant τ , find the radius Λ_{\min} of the smallest circle of curvature of $y(t)$, the radius being drawn in the direction of the time axis, so that Λ_{\min} is expressed in units of time. Let η_0 be the distance, in units of length on the graph paper, corresponding to the full-scale value of y ; let ξ_0 be the distance, in the same units of length, but in the time direction, corresponding to the τ of the system. Define a normalized radius of curvature

$$(7.13-3) \quad \Lambda_{\min}^* = \Lambda_{\min} \eta_0 / \xi_0 \quad .$$

Then fidelity of dynamic response is likely if Λ_{\min}^* is large compared to τ , and Eq. (7.13-2) may be used.

In the general case, if the response $y(t)$ is known as a continuous, differentiable function of time, the value of $Y(t)$ that produced the response is obtained by performing the operations indicated in Eq. (7.1-1):

- (1) At any point (t_i, y_i) , determine $(dy/dt)_i$.
- (2) Multiply $(dy/dt)_i$ by τ .
- (3) Add the product to y_i . This sum is $Y(t_i)$.

In particular, wherever $dy/dt = 0$, the quantities Y and y are equal (but their slopes are not).

The procedure is applicable even when τ varies with t or y , provided that τ is known. It is necessary only to use the appropriate value of τ in step 2. [Note N7.13]

The requirement of differentiability implies that the procedure may not be applied at sharp corners in $y(t)$.

7.14 Damping of a second-order system. Second-order systems fall into one of several classes that are distinguished by the value of ζ , the fraction of critical damping.

(1) If $0 < \zeta < 1$, the system is termed *underdamped* and is capable of an oscillatory transient response. Such a system is noticeably different from a first-order system, which is incapable of an oscillatory transient response.

(2) If $\zeta = 0$, the system is termed *undamped*. If disturbed, it would enter into an oscillation, at frequency f_n , that would never die out. [Note N7.14]

(3) If $\zeta > 1$, the system is termed *overdamped* and is incapable of an oscillatory transient response. As ζ increases beyond unity, the system more closely begins to resemble a first-order system.

(4) If $\zeta \gg 1$, the system behaves substantially like a first-order system.

(5) If $\zeta = 1$, the system is termed *critically damped*. The condition represents the transition between the underdamped and the overdamped states. The transition is quite gradual as far as the system's physical behavior is concerned, although there may be distinct changes in the mathematical descriptions of that behavior. The value $\zeta = 1$ is the smallest value of ζ at which transient oscillatory responses become impossible.

The behavior of an underdamped system is usually best described in terms of the parameters T_n , f_n , ω_n . The behavior of an overdamped system is usually best described in terms of τ . The relationship among these parameters is given by Eq. (7.1-3).

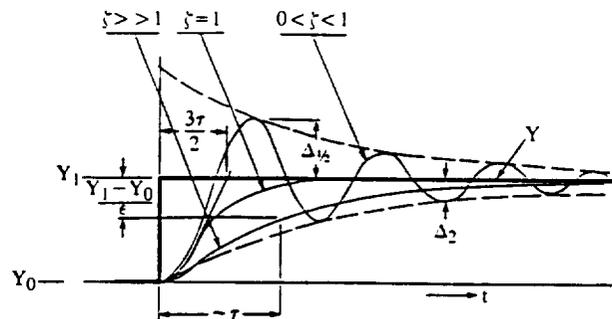


Figure 7.15.—Response of a second-order system to a step change.

7.15 Response of a second-order system to a step change. (Fig. 7.15) If

$$(7.15-1a) \quad Y - Y_0 = Y_1 \quad \text{for } t > 0$$

$$(7.15-1b) \quad Y = y = Y_0; \quad \dot{y} = 0 \quad \text{for } t = 0$$

then

(A) If $\zeta = 0$, y will start out at zero slope, approach Y_1 , overshoot, and then oscillate sinusoidally about Y_1 with amplitude $(Y_1 - Y_0)$ and frequency f_n .

(B) If $0 < \zeta < 1$, y will start out at zero slope, approach Y_1 , overshoot, and then oscillate about Y_1 with continually decreasing amplitude. [Note N7.15.1]

If the amplitude of the first overshoot beyond Y_1 is denoted by $\Delta_{1/2}$ (Fig. 7.15) and succeeding overshoots by Δ_n , where n denotes an integral number of complete oscillations if n is an integer, these oscillations have the following characteristics:

- (1) The period of the oscillations is T_n/α
where

$$(7.15-2) \quad \alpha = \sqrt{|1 - \zeta^2|} \quad .$$

- (2) The logarithm of the ratio (> 1) of the amplitudes of two oscillations m cycles apart is given by

$$(7.15-3) \quad \ln(\Delta_n/\Delta_{n+m}) = m \cdot \delta$$

where

$$(7.15-4) \quad \delta = 2\pi\zeta/\alpha$$

is the *logarithmic decrement*. It is the logarithm of the ratio of the amplitudes of successive cycles.
[Note N7.15.2]

- (3) The envelopes of the positive and negative peaks of the oscillations are a pair of symmetrical, approximately exponential curves having apparent time constants

$$(7.15-5) \quad \tau/(2\zeta^2) = T_n/(2\pi\zeta) \quad .$$

- (4) If $\zeta < 0.6$, the time taken for $|y - Y_1|$ to become less than 10, 5, or 2 percent of $|Y_0 - Y_1|$ is approximately $T_n/(\pi\zeta)$, $1.5 T_n/(\pi\zeta)$, or $2T_n/(\pi\zeta)$, respectively.
 (5) Qualitatively, the following characteristics of the oscillations are noteworthy:
 (a) The period of the oscillations increases as ζ increases.
 (b) The decay of the oscillations is more rapid as ζ increases.
 (c) The time for $|y - Y_1|$ to become less than a given small percentage of the initial step $|Y_0 - Y_1|$ becomes shorter as ζ increases.

(C) If ζ is increased to unity, y will start out at zero slope, pass through a point of inflection (t_2, y_2) , and thereafter approach Y_1 in the manner of an exponential curve with time constant slightly higher than τ . The response is aperiodic, with no overshoot.

The excursion from Y_0 to Y_1 will be almost 99 percent complete after a time interval T_n . [Note N7.15.3]
This time interval is shorter than it would be for any other value of ζ .

(D) If ζ is increased beyond the value of unity, the response will start out at zero slope, pass through a point of inflection (t_2, y_2) , and thereafter approach Y_1 in the manner of an exponential curve with time constant slightly higher than τ . The value of t_2 decreases (from $\tau/2$ at $\zeta = 1$) as ζ increases, and approaches zero as $\zeta \rightarrow \infty$. The value of y_2 approaches Y_0 as $\zeta \rightarrow \infty$. [Note N7.15.4]

Practically, for $\zeta \geq 1$, the response is a simple exponential function with time constant τ , once the inflection point has been passed. [Note N7.15.5] Consequently, if $\zeta \gg 1$, the response is substantially that of the basic, first-order system with time constant τ .

7.16 Response of a second-order system to an exponential change. (Fig. 7.16) The response treated in this section is representative of the behavior of

- (i) a galvanometer connected to a thermocouple, or
- (ii) a *probe microphone* wherein a microphone or pressure gauge must be placed at the far end of a tube because of space limitations or environmental considerations.

If the forcing function is

$$(7.16-1a) \quad Y = Y_1 - (Y_1 - Y_0)e^{-t/\lambda} \quad \text{for } t \geq 0$$

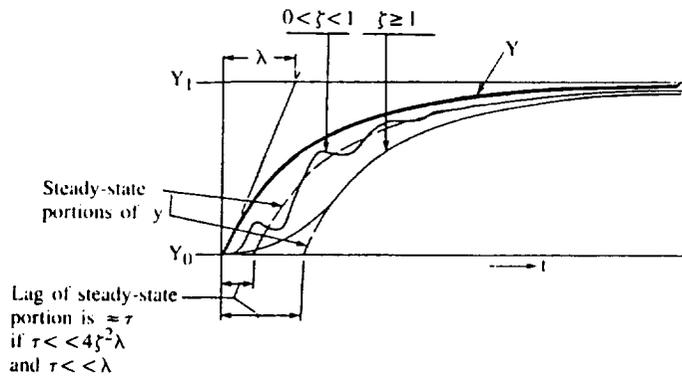


Figure 7.16.—Response of a second-order system to an exponential forcing function.

and

$$(7.16-1b) \quad Y = y = Y_0 ; \quad \dot{y} = 0 \quad \text{for } t = 0$$

then y will start out at zero slope and approach Y in one of two ways:

A. If $\zeta < 1$, the response will be oscillatory; if $\zeta \neq 0$, the oscillation amplitude will decrease with time.

B. If $\zeta \geq 1$, the response will be aperiodic and monotonic, and will approximate an exponential shape after it has passed through a point of inflection.

(i) If $(\lambda/\tau) \gg 4\zeta^2$, the time required for the transient to die out, for all practical purposes, is short compared to λ . The response y will then approach a steady-state condition in which it lags Y by a constant time interval given by

$$(7.16-2) \quad \text{time lag} = -\lambda \ln[1 - (\tau/\lambda) + \tau^2/(4\zeta^2\lambda^2)] \\ \approx -\lambda \ln(1 - \tau/\lambda)$$

(ii) If $(\lambda/\tau) \gg 4\zeta^2$ and, in addition, $\lambda \gg \tau$, this time lag is substantially equal to τ and the later portion of the response is similar to that of a first-order system, as shown in Fig. 7.8.

(iii) If $(\lambda/\tau) \ll 4\zeta^2$ and $\lambda \ll \tau$, the response y will be similar to the response to the step change treated in Sec. 7.15 except that the response will be displaced by an additional time lag whose magnitude is on the order of λ . [Note N7.16]

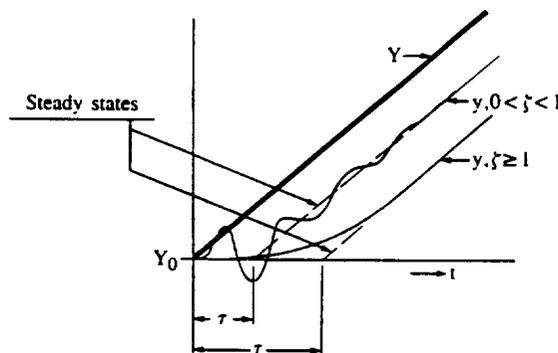


Figure 7.17.—Response of a second-order system to a ramp.

7.17 Response of a second-order system to a ramp. (Fig. 7.17) If

$$(7.17-1a) \quad Y = Y_0 + at \quad \text{for } t > 0$$

$$(7.17-1b) \quad Y = y = Y_0; \quad \dot{y} = 0 \quad \text{for } t = 0$$

then the response y will start out at zero slope and approach the steady-state condition

$$(7.17-2) \quad y = Y_0 + a(t - \tau)$$

Thus, the asymptote is a ramp parallel to the forcing function but lagging behind it by the constant amount τ .

The approach to the steady state is through a transient that is oscillatory if $\zeta < 1$ and is aperiodic and monotonic if $\zeta \geq 1$. The complete response is given by

$$(7.17-3) \quad y = Y_0 + a[t - \tau(1 - A)]$$

where A is a dimensionless function of ζ and of t/τ (or t/T_n) and represents the fractional amount by which the time lag differs from τ .

(i) If $0.2 \leq \zeta < 1$, the value of A is less than 0.10, 0.05, or 0.02 after a time $T_n/(2\zeta)$, $2T_n/(3\zeta)$, or $5T_n/(6\zeta)$, respectively.

(ii) If $\zeta \geq 1$, the value of A is less than 0.10, 0.05, or 0.02 after a time 2.3τ , 3τ , or 4τ , respectively.
[Note N7.17]

The response treated in this section is representative of the behavior of a pressure gauge or manometer used to make a steady, continuous traverse of a region in which there is a linear pressure gradient.

Example 7.17 Suppose that a pressure gradient is to be surveyed with a critically damped pressure gauge having a time constant of τ seconds, to be moved at a velocity V . The inaccuracy of pressure indication p due to the method of surveying is not to exceed 0.1 percent. The *a priori* estimate of the magnitude of the pressure gradient is 1 percent/cm.

Then the value of a in Eq. (7.17-1) is

$$a = (0.01 p/\text{cm})V$$

The steady-state correction is

$$a\tau = (0.01 p/\text{cm})V\tau$$

If this correction is to be acceptably small it is necessary that

$$(0.01 p/\text{cm})V\tau < 0.001 p$$

or

$$V < (0.1/\tau) \text{ cm/s}$$

If this traverse velocity is unacceptably low, a higher traverse velocity may be used, provided that

- (1) the true pressure at time t is taken as the pressure indicated at time $(t + \tau)$, and
- (2) at least about 3τ seconds have elapsed after the initiation of the traverse.

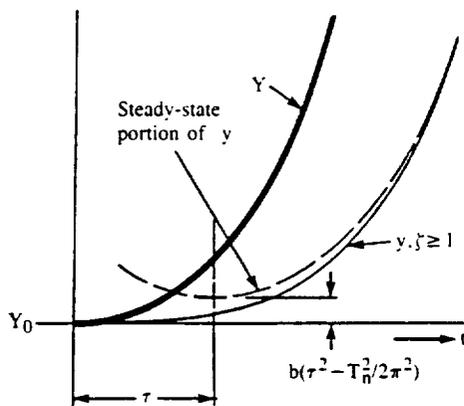


Figure 7.18.—Response of a second-order system to a quadratic forcing function.

7.18 Response of a second-order system to a quadratic forcing function. (Fig. 7.18) If

$$(7.18-1a) \quad Y = Y_0 + bt^2 \quad \text{for } t \geq 0$$

$$(7.18-1b) \quad Y = y = Y_0; \quad \dot{y} = 0 \quad \text{for } t = 0$$

the response y will start out at zero slope and approach a steady state represented by a parabola having the same latus rectum as the forcing function but with vertex displaced to the coordinates (τ, y_b) , where

$$(7.18-2) \quad y_b = b(\tau^2 - 2/\omega_n^2) = b[\tau^2 - T_n^2/(2\pi^2)] \quad .$$

The steady-state condition is represented by

$$(7.18-3) \quad \text{steady-state } y = b(t - \tau)^2 + b\tau^2[1 - (2\zeta^2)^{-1}] \quad .$$

This steady-state response resembles that of the first-order system (Sec. 7.10), differing from it only in the position of the vertex, which no longer lies on the original forcing function but is displaced from it by a vertical distance $b\tau^2/(2\zeta^2)$.

If $\zeta = 1/\sqrt{2}$, the steady-state y lags behind Y by a constant time interval equal to τ .

If $\zeta \neq 1/\sqrt{2}$, the time lag approaches τ as time increases.

The approach to the steady state is oscillatory if $\zeta < 1$ and is aperiodic if $\zeta > 1$. [Note N7.18]

If $0.2 < \zeta < 0.8$, the time when the response will differ from the steady-state value by less than 10, 5, or 2 percent of that value will be on the order of $T_n/2$, $5T_n/8$, or $3T_n/4$, respectively. If $\zeta \geq 0.8$, the corresponding time will not exceed 2.2τ , 2.5τ , or 3.0τ , respectively.

The response treated in this section is representative of the behavior of an impact-plate flowmeter (wherein the dynamic pressure of a fluid is converted into a mechanical force) or of a dynamic pressure transducer used to make a steady continuous traverse of a region in which there is a linear fluid-velocity gradient.

7.19 Response of a second-order system to an impulse. If a second-order system is subjected to a step change of very high magnitude and very short duration such that the product of duration and magnitude is a finite quantity I , termed the *impulse*, then the response will reach a maximum value in a time dependent on the natural period of the system and then drop back to its initial value in a manner identical to the response to a step change.

The behavior is representative of the ballistic galvanometer, which is usually used in the critically damped condition, or of the ballistic pendulum. The latter device is convenient for measuring the impulse of an explosive charge or of a pulsed rocket.

If

(1) the system is initially such that

$$(7.19-1) \quad y = Y_0; \dot{y} = 0 \quad ,$$

(2) the pulse duration Δt is so short that

$$(7.19-2a) \quad \Delta t \ll T_n/(2\pi) \quad \text{if} \quad \zeta \leq 1$$

$$(7.19-2b) \quad \Delta t \ll \zeta T_n/(2\pi) \quad \text{if} \quad \zeta \geq 1 \quad ,$$

(3) the pulse amplitude $(Y_1 - Y_0)$ is so large that

$$(7.19-3) \quad I = (Y_1 - Y_0) \bullet \Delta t \quad ,$$

then

(1) the response will reach a maximum value y_m given by

$$(7.19-4a) \quad y_m - Y_0 = I\omega_n \exp[-(\zeta/\alpha)\sin^{-1}\alpha] \quad \text{if} \quad \zeta < 1$$

$$(7.19-4b) \quad y_m - Y_0 = I\omega_n \epsilon^{-1} \quad \text{if} \quad \zeta = 1$$

$$(7.19-4c) \quad y_m - Y_0 = I\omega_n \exp[-(\zeta/\alpha)\sinh^{-1}\alpha] \quad \text{if} \quad \zeta > 1 \quad .$$

(2) This maximum value will be reached at a time t_m given by

$$(7.19-5a) \quad t_m \approx (T_n/2\pi)\zeta^{-1} \quad \text{if} \quad \zeta \leq 1$$

$$(7.19-5b) \quad t_m \approx (T_n/2\pi)(\sinh^{-1}\alpha)/\alpha \\ \approx (T_n/2\pi)\zeta^{-1/2} \quad \text{if} \quad \zeta > 1 \quad .$$

(3) Thereafter the step-change response equations of Sec. 7.15 will apply, except that t in those equations must be replaced by $(t - t_m)$. Amplitude ratios and oscillation periods, when $\zeta < 1$, remain as in Sec. 7.15. [Note N7.19]

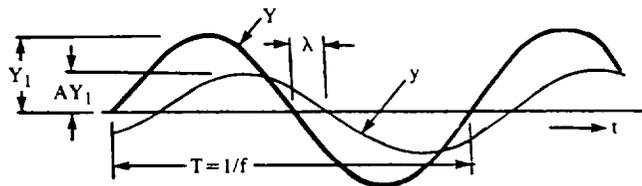


Figure 7.20.1.—Steady-state response of a second-order system to a sinusoid.

7.20 Response of a second-order system to a sinusoid. Only the steady-state response is worthy of consideration here [Note N7.20.1]. The application of a steady, sinusoidal forcing function

$$(7.20-1) \quad Y = Y_1 \sin(\omega t) = Y_1 \sin(2\pi f t) = Y_1 \sin(2\pi t/T)$$

will result in a sinusoidal response of the same frequency but with different amplitude and with lag in time and phase (Fig. 7.20.1). This response is represented by

$$(7.20-2a) \quad y = AY_1 \sin(\omega t - \varphi)$$

$$(7.20-2b) \quad = AY_1 \sin[\omega(t - \lambda)]$$

where

$$(7.20-2c) \quad A = [(1 - r^2)^2 + 4\zeta^2 r^2]^{-1/2}$$

$$(7.20-2d) \quad \varphi = \tan^{-1}[2\zeta r / (1 - r^2)]$$

$$(7.20-2e) \quad \lambda = \varphi / \omega = \varphi \cdot T / (2\pi)$$

$$(7.20-2f) \quad r = \omega / \omega_n = f / f_n = T_n / T$$

The quantities of interest are

the frequency ratio r ,

the amplitude ratio A ,

the phase-angle φ ,

the time lag λ .

The quantities A and φ , as functions of ζ and r , are shown in rectangular coordinates in Fig. 7.20.2, and in polar coordinates in Fig. 7.20.3.

The following features of the response are distinctive:

A. If $\zeta = 0$, the value of A is

$$(7.20-3) \quad A = (1 - r^2)^{-1}$$

On rectangular coordinates, $A(r)$ resembles a quadratic parabola, when $r \ll 1$. The amplitude ratio becomes infinite when $r = 1$. [Note N7.20.2]

B. If $0 < \zeta < 1/\sqrt{2}$,

(i) the amplitude ratio A has a maximum at a frequency f_r , termed the *resonant frequency* and given by

$$(7.20-4) \quad f_r = f_n \sqrt{1 - 2\zeta^2}$$

(ii) the magnitude A_r of the amplitude ratio at the resonant frequency is

$$(7.20-5) \quad A_r = (2\zeta\alpha)^{-1}$$

where

$$(7.20-6) \quad \alpha = \sqrt{|1 - \zeta^2|}$$

(iii) the locus of these maxima for all values of $0 < \zeta < 1/\sqrt{2}$ is given by the curve

$$(7.20-7) \quad A_r = (1 - r^4)^{-1/2}$$

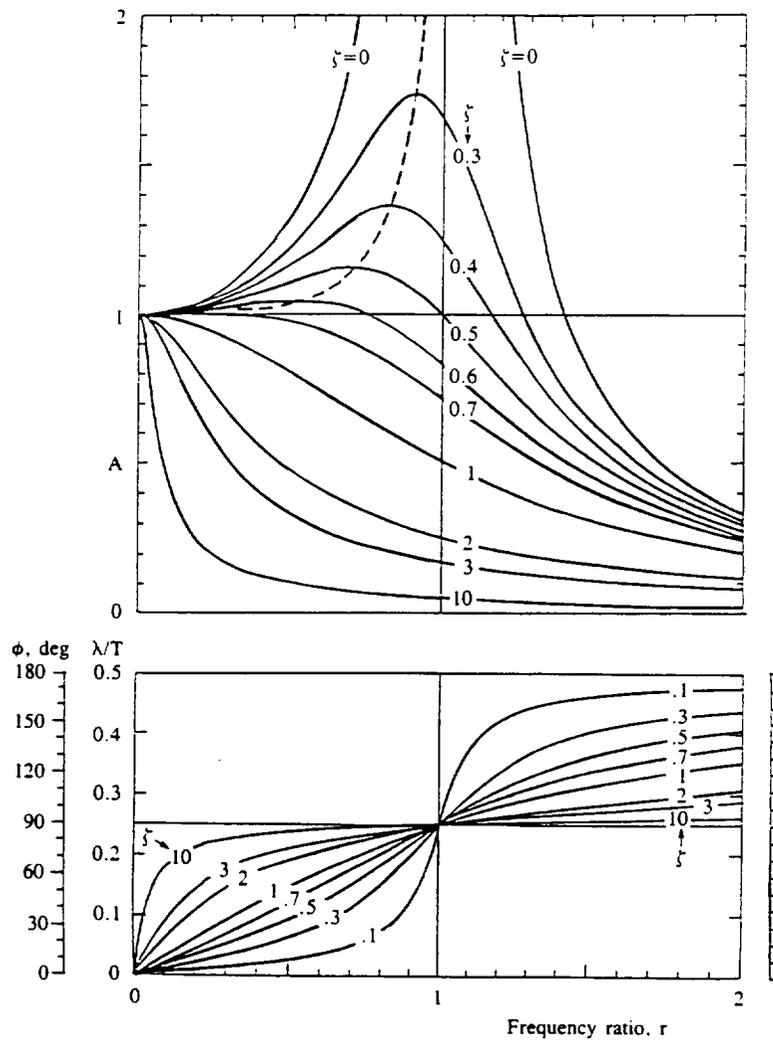


Figure 7.20.2—Amplitude ratio A , time lag λ , and phase angle ϕ of a second-order system's response to a sinusoid.

(iv) for any one curve for which $\zeta < 0.2$, there are two values of r at which $A - 1 = (A_r - 1)/2$. The difference $\Delta r_{1/2}$ between these two values is termed the "bandwidth at half maximum" [Note N7.20.3]. Its relation to ζ is given by

$$(7.20-8) \quad \zeta \approx \Delta r_{1/2} / [2\sqrt{3}(1 - 0.84\Delta r_{1/2})] \quad \Delta r_{1/2} < 0.4$$

(v) if "fidelity" of response implies that A shall be near to unity, this condition is more likely to be approached over a very wide range of r when $0.5 \leq \zeta \leq 0.6$. Thus:

$$0.85 < A < 1.15 \quad \text{for} \quad 0 < r < 1.14 \quad \text{when} \quad \zeta = 0.5$$

$$0.95 < A < 1.05 \quad \text{for} \quad 0 < r < 0.85 \quad \text{when} \quad \zeta = 0.6$$

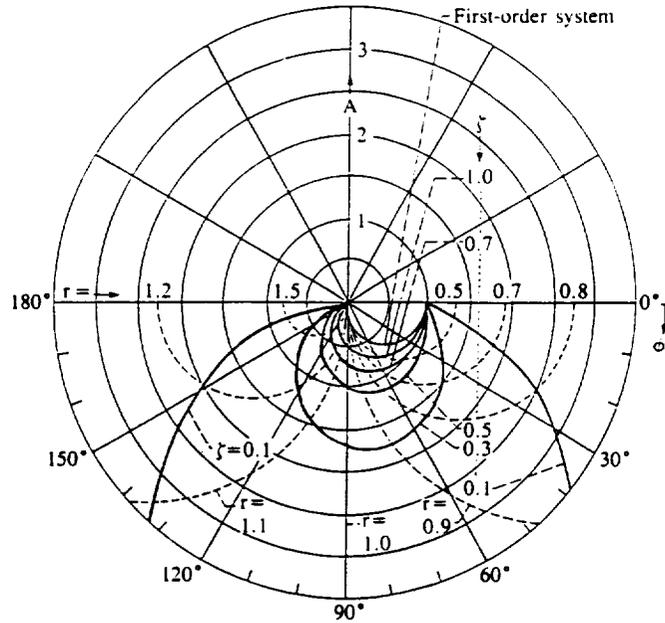


Figure 7.20.3.—Amplitude ratio A versus phase angle ϕ of response to a sinusoid by a second-order system.

If $\zeta = 1/\sqrt{2}$, then both $dA/dr = 0$ and $d^2A/dr^2 = 0$ at $r = 0$, implying maximum fidelity of response when $r \ll 1$. However, there is a 5-percent drop at $r = 0.58$ and a 15-percent drop at $r = 0.79$.

Thus, for any preassigned limit of error in amplitude response there may be found an optimum value of ζ such that the error remains within the limit over the widest possible range of r . However, the selection of ζ must also be guided by the fact that a random uncertainty in ζ also exists, either because of imperfect knowledge or because of imperfect control. Therefore, in practical situations, any curve in Fig. 7.20.2 must be replaced by a band representing the uncertainty in knowledge or control of ζ .

In the extreme case when ζ is unknown except for the knowledge that the system is underdamped, then

$$(7.20-9) \quad (1 + r^2)^{-1} < A < (1 - r^2)^{-1} .$$

The left-hand limit represents the case of critical damping, $\zeta = 1$; the right-hand limit represents the case of no damping, $\zeta = 0$.

C. When $\zeta > 1/\sqrt{2}$, there is no resonant frequency. The response drops monotonically as r increases but there is an inflection point.¹

The drop with increasing r is more abrupt as ζ increases. As $\zeta \rightarrow \infty$, the response approaches that of a first-order system.

D. For all values of ζ , the lagging phase angle ϕ changes from 0 at $r = 0$, to $\pi/2$ at $r = 1$, and approaches π as $r \rightarrow \infty$. The value of ζ affects the manner of transition between these fixed points or limits.

For all values of ζ , the time lag λ changes from 0 at $r = 0$, to $T/4$ at $r = 1$, and approaches $T/2$ as $r \rightarrow \infty$. The value of ζ affects the manner of transition between these fixed points.

For $\zeta = 1/\sqrt{2}$, both phase lag ϕ and the ratio λ/T vary virtually linearly with r over the range $0 < r < 1$. Consequently, there will be little or negligible distortion of a complex wave shape.

¹The inflection point occurs at $r = 1$ when $\zeta = 1/\sqrt{2}$, at $r = 1/\sqrt{3}$ when $\zeta = 1$, and very nearly at $r = 1/[2\sqrt{(2\zeta^2 - 1)}]$ when $\zeta \geq 2$.

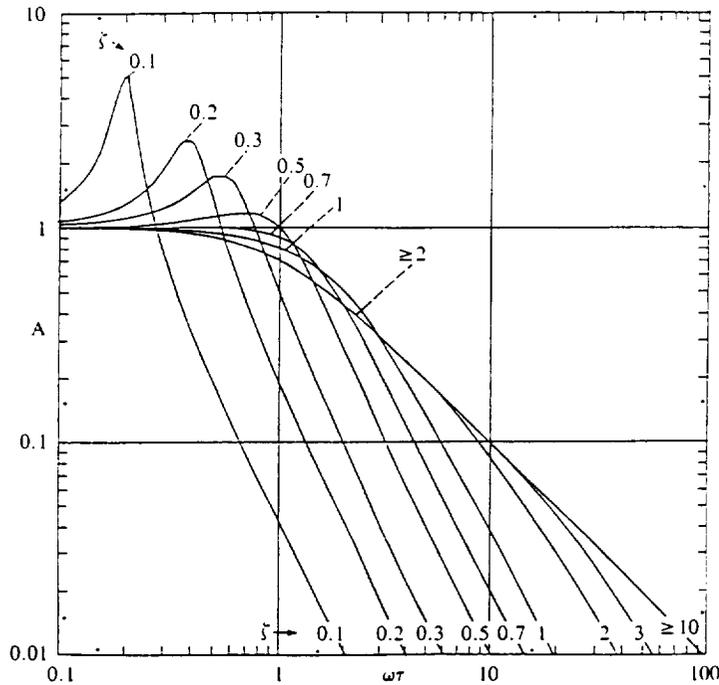


Figure 7.20.4.—Amplitude ratio of a second-order system's response to a sinusoid.

E. The use of a logarithmic plot of A versus $\omega\tau$ further clarifies the nature of the response (Fig. 7.20.4).

(i) For underdamped systems, the line

$$(7.20-10) \quad A = 1/(\omega\tau)$$

is the locus of the maxima of the response curves. The abscissa of any maximum, from Eq. (7.20-4), is

$$(7.20-11) \quad \omega\tau = 2\zeta \sqrt{1 - 2\zeta^2}$$

The ordinate at the maximum is given by Eq. (7.20-5).

(ii) For underdamped systems, as $\omega\tau$ becomes much larger than unity, the slope of $A(\omega\tau)$ approaches -2 ("12 dB per octave").

(iii) For overdamped systems, if $\omega\tau$ becomes much larger than unity, but remains much larger than $4\zeta^2$, then $A(\omega\tau)$ becomes approximately equal to $4\zeta^2/(\omega\tau)^2$ so that the asymptotic slope of $A(\omega\tau)$ is -2 , the same as that of an underdamped system. However, if $\zeta \rightarrow \infty$ but $4\zeta^2 \gg \omega\tau \gg 1$, the system begins to resemble a first-order one, with $A(\omega\tau) \approx 1/(\omega\tau)$.

F. A polar plot with coordinates (A, φ) is shown in Fig. 7.20.3. All points lie in the first two negative quadrants [Note N7.20.4]. Contour lines of $\zeta = \text{constant}$ and $r = \text{constant}$ are shown.

Contours for $\zeta > 3$ are almost indistinguishable from the semicircle which represents the polar plot for a first-order system (Fig. 7.12.3).

Contours of $r = \text{constant}$ ($r \neq 1$) are semicircles passing through the origin and having their centers at $A = 0.5/|1 - r^2|$ and $\varphi = 0$ or π .

G. The amplitude of $\dot{y}/(\omega_n Y_1)$ is rA . This amplitude has a maximum value of $1/(2\zeta)$; the maximum occurs at $r = 1$, regardless of the value of ζ .

7.21 Response of a second-order system to any forcing function. If any arbitrary forcing function can be approximated as a sum of the forcing functions treated in Secs. 7.15 to 7.20, the steady-state solution is the sum of the steady-state solutions for the component forcing functions. However, the transient solution cannot be deduced by any simpler mathematical procedures than by the solution of Eq. 7.1-2.

The converse problem is of more common practical interest: given an instrument indication $y(t)$ of arbitrary wave shape, to deduce the forcing function $Y(t)$ that produced it. Theoretically, the same technique as described in Sec. 7.13 should be usable; one adds, to the indication $y(t)$, corrections that are proportional to the first and second derivatives of $y(t)$, in accordance with Eq. 7.1-2, thereby deriving the forcing function $Y(t)$. In practice, this correction procedure has rarely been effective, because of the combined effect of errors caused by the process of double differentiation, the uncertainty or inconstancy of ζ , and the presence of noise.

Under favorable circumstances, in a time interval $\pm \Delta t$ surrounding a particular value $y(t_x)$, the value of $Y(t_x)$ may be deducible from the information in Secs. 7.15 to 7.20. Such favorable circumstances may be

- (1) The instrument is sufficiently damped so that ζ exceeds 0.5.
- (2) The principal fluctuations occur at frequencies well below f_n .
- (3) Adequate time has elapsed for decay of the transient that originated when the forcing function was first applied.
- (4) Abrupt changes in the forcing function, as revealed by the recurrence of damped oscillations, are absent.

Under such circumstances, if the indication in the vicinity of $y(t_x)$ is oscillatory, the amplitude and time lag corrections of Sec. 7.20 may be applied to deduce $Y(t_x)$. If the indication in the vicinity described by $\Delta t \approx 3\tau$ appears monotonic, the assumption that there is a time lag of magnitude τ , as suggested by Secs. 7.17 and 7.18, may provide a closer estimate of $Y(t_x)$.

In the absence of identifiable oscillations, a test that will indicate whether $y(t)$ is likely to be a reasonably faithful representation of $Y(t)$ is provided by a procedure similar to that outlined in Sec. 7.13:

The radius Λ_{\min} (in units of time) of the smallest radius of curvature of $y(t)$ is determined, and the normalized radius of curvature

$$(7.21-1) \quad \Lambda_{\min}^* = \Lambda_{\min} \eta_0 / \xi_0$$

is computed, where η_0 is as defined in Sec. 7.13 but ξ_0 here represents the distance on the graph paper corresponding to T_n . Then fidelity of response is likely if Λ_{\min}^* is large compared to T_n .

If a dynamic system has little damping (say, $\zeta < 0.2$), the natural oscillations (at period close to T_n) that appear in $y(t)$ may obscure the representation of $Y(t)$ even when $Y(t)$ varies slowly. In such case, a curve may be drawn through the midpoint of every half cycle of the oscillations in $y(t)$, this median curve may be treated as an alternative "faired curve," and its Λ_{\min}^* may be determined. If $\Lambda_{\min}^* \gg T_n$, this faired $y(t)$ is likely to be a reasonably faithful representation of $Y(t)$.

7.22 Characterization of the order of a system. The basis upon which a system is designated "first order" or "second order" is purely a pragmatic one. If the behavior of a system is indistinguishable from that of a first-order system, it may be designated and treated as a first-order system, even though it is known to contain elements that, in a strict mathematical sense, make it a second-order system.

In many borderline cases, such as overdamped second-order systems, large savings in analytical or experimental effort may be realized by treating the system as one of the first order, with an acceptably small loss in accuracy.

Example 7.22a. Any real electrical circuit containing a resistor and a capacitor necessarily contains some residual inductance in the connecting wires and in the lumped elements. At lower frequencies, the inductive reactance may be kept negligible and its presence need not prevent consideration of the circuit as a first-order system. On the other hand, at sufficiently high frequencies, the inductive

reactance of even a high-quality capacitor may become a significantly large component of the total reactance, so that the system must be treated as a second-order one.

Example 7.22b. A thermometer bulb with distributed mass and finite thermal conductivity constitutes a system of higher than first order. In fact, at sufficiently high frequencies of temperature fluctuation, the system must be described by the partial differential equation (Fourier's equation) of heat conduction, which represents the limiting condition of a series of first-order systems in tandem, each subjected to the forcing function generated by its predecessor. Nevertheless, for many designs and applications, no appreciable error is made by treating the bulb as having lumped mass and infinite thermal conductivity, and hence as being a first-order system.

7.23 Instrument selection to provide discrimination in frequency response. If an instrument is to be used to indicate variations in the magnitude of a forcing function $Y(t)$, the indications may be obscured by the presence of variations in $Y(t)$ that occur at frequencies much higher than the frequencies of the variations that are of principal interest [Note N7.23]. It is therefore desirable to select an instrument that

- (i) will provide reliable indications of the magnitude of low-frequency components of the input signal, and
- (ii) will have minimal response to high-frequency components of the input signal.

Solution of this problem requires the instrument user's intuitive perception, *a priori*, of

- (1) what frequencies might be present in the signal $Y(t)$,
- (2) what frequencies are of principal interest in the current application of the instrument, and
- (3) what frequencies are not of interest.

If frequencies in category 3 are much higher than those in category 2, it may be possible to select an instrument whose response at the higher frequencies is much poorer than the response at the lower frequencies.

Generally, the optimum choices are these:

- (a) If the measuring instrument is essentially a first-order system, its time constant τ should not be much shorter than the value that would just produce an acceptable measurement error.
- (b) If the measuring instrument is a second-order system, its fraction of critical damping ζ should lie in the vicinity of 0.6 to 0.8 and its natural frequency f_n should be not much higher than the value that would produce an acceptable error at the highest frequency f that is of principal interest.

The need for these optimum choices may not be important when

- (i) high-frequency components of $Y(t)$ are not so large as to obscure materially the components of principal interest, or
- (ii) appropriate reduction (filtering) of high-frequency indications can be performed during the data-reduction process.

At the other extreme, the optimum choices indicated above are essential

- (i) when an almost undamped mechanical instrument may be subjected to frequencies so near to its resonant frequency that response amplitudes will be destructively large or so large that the calibration or elastic properties (Sec. 5.6) of the instrument will be impaired. (Subsequent electrical filtering cannot prevent such impairment.)
- (ii) when an almost undamped electrical circuit may be subjected to frequencies so near to its resonant frequency that saturation of amplifiers may impair the ability to represent correctly the amplitudes at the frequencies of principal interest.

Example 7.23a. Structural loads developed in airplane landing gear are often studied by measuring the acceleration of the airplane, treated as a single lumped mass, by use of an accelerometer mounted near the center of gravity (c.g.) of the airplane. The c.g. acceleration is usually less than 1 g ($1 \text{ g} \approx 10 \text{ m/s}^2$) and is imposed at an equivalent frequency on the order of 1 Hz. Any accelerometer attached to the airplane structure may also be subjected to local structural vibrations on the order of 1 g, but at frequencies of 20 to 1000 Hz. The measurement of interest would therefore be impaired severely if the accelerometer had high fidelity at all frequencies. However, an accelerometer with $\zeta = 0.7 \pm 0.1$ and $f_n \approx 5 \text{ Hz}$ would provide more than a 10-fold attenuation in indication at high frequencies and less than 1-percent error at 1 Hz.

Example 7.23b. A 10-Hz signal derived from a 10 000-ohm source is to be observed on a cathode-ray oscilloscope, when a 60-Hz hum signal of comparable magnitude is superposed on the 10-Hz signal. A 1- μ F capacitor across the input terminals of the oscilloscope will produce a 15-percent systematic reduction in the desired signal but will produce a 4-fold attenuation of the hum. Application of a correction factor of 1.15 to the indicated 10-Hz signal may then result in an acceptably small uncertainty in the measurement.

7.24 Initial study of a dynamically changing phenomenon. When a new, unfamiliar experiment or test is undertaken, an appropriate first step is to obtain a continuous oscillographic record of the variation with time of the principal physical variables that are of concern. High measurement accuracy is not necessary, but the measurement system should have reasonably good fidelity in the frequency band that is of interest. (A response like that of a slightly underdamped second-order system would be appropriate.) Visual inspection of the record will then reveal signal frequencies and rates of change that are present, and their relative magnitudes. One may then determine whether there are present, in significant magnitudes,

- (1) power-line frequencies, or their multiples, that may be attributable to hum;
- (2) oscillation frequencies that may be attributable to slightly damped instruments or transmission systems;
- (3) oscillations, at higher frequencies outside of the band of interest, that are large enough to obscure important wave patterns at lower frequencies;
- (4) spikes or pulses so large that there is danger of damage to instruments or to the validity of acquired data;
- (5) signal frequencies truly representative of the phenomenon to be studied.

An indication of the probable wave shape of the signal will also be obtained. This diagnosis can then guide the application of appropriate remedial measures that appear desirable (including limitation of the frequency-response band), and can provide a clearer *a priori* definition of static and dynamic accuracies that appear necessary in subsequent experiments or tests.

NOTES FOR CHAPTER 7.

N7.4 The clearest way to describe the technique of conversion of a higher-order differential equation to a combination of first-order and second-order equations is to assign the symbol p to the operator d/dt and to treat p as an algebraic quantity, so that

$$(N7.4-1) \quad d^n y/dt^n = p^n y \quad .$$

Thus, Eq. (7.1-1) may be written as

$$(N7.4-2) \quad \tau \bullet p y + y = Y$$

so that, symbolically, the solution for y may be written as

$$(N7.4-3) \quad y = Y/(\tau p + 1) \quad .$$

Similarly, Eq. (7.1-2c) may be written as

$$(N7.4-4a) \quad [(1/\omega_n^2)p^2 + (2\zeta/\omega_n)p + 1]y = Y \quad .$$

Here, there are two cases to consider:

- (1) If $\zeta \geq 1$, the equation

$$(N7.4-5) \quad (1/\omega_n^2)p^2 + (2\zeta/\omega_n)p + 1 = 0$$

has $\zeta > 0$, positive, real roots (if $\zeta = 1$, the two roots happen to be equal). If these roots are designated $1/\tau_1$ and $1/\tau_2$, Eq. (N7.4-4a) may be written as

$$(N7.4-4b) \quad (\tau_1 p + 1)(\tau_2 p + 1)y = Y$$

so that, symbolically, one may write

$$(N7.4-4c) \quad (\tau_2 p + 1)y = Y/(\tau_1 p + 1)$$

or

$$(N7.4-4d) \quad y = \frac{1}{\tau_2 p + 1} \cdot \frac{1}{\tau_1 p + 1} \cdot Y \quad .$$

Consequently, in this case, solution of the second-order Eq. (N7.4-4a) may be obtained by first finding the response y_1 of a first-order system with time constant τ_1 to a forcing function Y and then finding the response y of a first-order system with time constant τ_2 to a forcing function y_1 . The order in which the quantities $(\tau_1 p + 1)$ and $(\tau_2 p + 1)$ appear is immaterial.

(2) If $\zeta < 1$, Eq. (N7.4-5) has two conjugate, complex roots. A formulation in terms of real parameters like τ_1 and τ_2 is not possible, and one can only write

$$(N7.4-6) \quad y = Y/[(1/\omega_n^2)p^2 + (2\zeta/\omega_n)p + 1] \quad .$$

Turning now to the higher-order differential equation which may be written symbolically as

$$(N7.4-7) \quad (a_n p^n + a_{n-1} p^{n-1} + \dots + a_2 p^2 + a_1 p + 1)y = Y \quad ,$$

we note that the equation

$$(N7.4-8a) \quad a_n p^n + a_{n-1} p^{n-1} + \dots + a_2 p^2 + a_1 p + 1 = 0$$

may have k real roots and m pairs of conjugate complex roots, such that

$$(N7.4-8b) \quad k + 2m = n \quad .$$

Equation (N7.4-7) may then be factored and written as

$$(N7.4-9a) \quad \Pi_1 \Pi_2 y = Y$$

where Π_1 is the product of factors F_i of the form

$$(N7.4-10) \quad F_i = (\tau_i p + 1) \quad (i = 1, 2, \dots, k)$$

and Π_2 is the product of factors G_j of the form

$$(N7.4-11) \quad G_j = p^2/\omega_{nj}^2 + (2\zeta_j/\omega_{nj})p + 1 \quad (j = 1, 2, \dots, m)$$

so that

$$(N7.4-9b) \quad \Pi_1 = \prod_1^k F_i; \quad \Pi_2 = \prod_1^m G_j \quad .$$

Thus, the solution of Eq. (N7.4-7) may be represented symbolically as

$$(N7.4-12) \quad y = \frac{1}{G_1} \cdot \frac{1}{G_2} \cdots \frac{1}{G_m} \cdot \frac{1}{F_1} \cdot \frac{1}{F_2} \cdots \frac{1}{F_k} Y$$

where the order of terms in the product is immaterial. This represents a tandem or serial sequence of operations.

Equation (N7.4-12) may also be written as

$$(N7.4-13) \quad y = \left(\frac{\beta_1}{G_1} + \frac{\beta_2}{G_2} + \dots + \frac{\beta_m}{G_m} + \frac{\alpha_1}{F_1} + \frac{\alpha_2}{F_2} + \dots + \frac{\alpha_k}{F_k} \right) Y$$

where the α 's are algebraic functions of the τ_i 's, the β 's are algebraic functions of p , the ζ_j 's, and the ω_{nj} 's, and where the order of terms in the sum is immaterial. This represents a parallel group of operations. Because of the presence of p in the β 's, this equation does not lend itself to an intuitive perception of system behavior, or to easy simulation by analog methods, unless Π_2 is unity (i.e., unless all the roots of Eq. (N7.4-8a) are real).

The use of the operator p , which is treated in most elementary texts on differential equations (where the preferred symbol is usually D), forms the basis of Heaviside's operational calculus. For easier mathematical solution of differential equations, the modern approach is the use of the Laplace transform (Ref. 7-1). However, the Laplace transform does not provide a *description* of the physical situation that permits easy interpretation and deduction of the system's response merely by inspection. The differential equation itself, or the symbolic use of the operator p or D , does provide such understanding more easily. This point has been made by Trimmer (Ref. 7-2), who also provides a more complete exposition of the subject of dynamic response.

N7.7 Equation (7.1-1), with right-hand term zero, occurs in the description of radioactivity. When a given mass of a radioactive nuclide emits particles, the mass becomes depleted. Since the number of particles emitted per unit time is proportional to the remaining mass, one may write

$$\Delta \dot{N} = - (1/\tau) \dot{N} \Delta t$$

to indicate that the drop $\Delta \dot{N}$ in emission rate will be proportional to the elapsed time Δt . If Δt is so large that the random error in measuring \dot{N} is acceptably small and if the measurements are recorded over a sufficiently long total time scale that the concepts of the differential calculus may be applied, it may be justifiable to write

$$\tau \bullet (d\dot{N}/dt) + \dot{N} = 0 \quad .$$

The *half-life* $\tau_{1/2}$ of the particular disintegration is the time interval in which N would become one half of its value at the beginning of the interval. Its relation to τ is

$$\tau_{1/2} = \tau \ln 2 \quad .$$

N7.8 As indicated in Sec. 7.4, the response y may be considered either (a) as the response of an instrument with time constant τ to an exponential forcing function with time constant λ or (b) as the response of an instrument with time constant λ to an exponential forcing function with time constant τ .

If λ is chosen as the larger of the two time constants, and we define

$$(N7.8-1) \quad m = \lambda/\tau \quad m \geq 1 \quad ,$$

the following relations, illustrated in Fig. 7.8, serve as convenient indicators of the nature of the response.

(1) The coordinates of the inflection point are given by

$$(N7.8-2a) \quad t_2 = (\lambda \ln m) / (m-1) \quad \lambda > \tau$$

$$(N7.8-2b) \quad t_2 = \lambda \quad \lambda = \tau$$

$$(N7.8-3a) \quad (y_2 - Y_0) / (Y_1 - Y_0) = 1 - (m + 1)m^{-m/(m-1)} \quad \lambda \geq \tau$$

$$(N7.8-3b) \quad \approx 0.26/m^{1/4}$$

(2) The tangent at the inflection point intersects $y = Y_0$ at t_1 , where

$$(N7.8-4) \quad t_1 \approx \tau m / (m + 3) \quad m \equiv (\lambda / \tau) \geq 1$$

(3) Over the range of possible values of λ/τ , the value of $y - Y_0$ at $t = \lambda + \tau$ will lie between 37 and 41 percent of $Y_1 - Y_0$:

(4) At times $t = n(\lambda + \tau)$ the value of $y - Y_0$ will be not less than the fraction of $Y_1 - Y_0$ tabulated below:

n	=	2	3	4	5
fraction	=	0.86	0.95	0.98	0.993

N7.9 If

$$(N7.9-1a) \quad Y = Y_1 + at \quad \text{for} \quad t > 0$$

$$(N7.9-1b) \quad Y = Y_0; \quad y = Y_0 \quad \text{at} \quad t = 0$$

then the change in Y is a combination of a step change of magnitude $(Y_1 - Y_0)$ and a ramp $Y_0 + at$. The response is

$$(N7.9-2) \quad y = Y_1 + a(t - \tau) + (Y_0 - Y_1 + a\tau)e^{-t/\tau}$$

and the asymptotic steady state is

$$(N7.9-3) \quad Y_1 + a(t - \tau).$$

N7.11 A pulse of triangular shape and very short duration Δt may be defined by the equations

$$(N7.11-1a) \quad Y - Y_0 = 2Y_1 t / \Delta t \quad \text{for} \quad 0 < t \leq \Delta t / 2$$

$$(N7.11-1b) \quad Y - Y_0 = 2Y_1 (1 - t / \Delta t) \quad \text{for} \quad (\Delta t / 2) \leq t \leq \Delta t \ll \tau$$

$$(N7.11-1c) \quad y = Y = Y_0 \quad \text{at} \quad t = 0$$

$$(N7.11-1d) \quad Y = Y_0 \quad \text{for} \quad t > \Delta t$$

$$(N7.11-2) \quad (Y_1 - Y_0) \cdot \Delta t / 2 = I$$

An approximation of the response is given by

$$(N7.11-3a) \quad y - Y_0 = (I/2\tau)[1 - \cos(\pi t/\Delta t)] \quad \text{for} \quad 0 \leq t \leq \Delta t$$

$$(N7.11-3b) \quad y - Y_0 = (I/\tau)\epsilon^{-t/\tau} \quad \text{for} \quad t > \Delta t$$

The response closely resembles the response to a rectangular pulse with the same value of I —the peak amplitude reached is $Y_0 + I/\tau$ and reversal of direction occurs only when $y = Y_0$ on the descending leg of the triangle, at which time t is almost equal to Δt .

N7.12.1 If the forcing function is a pure sinusoid of amplitude Y_1 and average value Y_0 and the initial value of y is y_0 , the magnitude of the transient will usually have become less than 15, 5, or 2 percent of $|Y_1 + Y_0 - y_0|$ at a time 2τ , 3τ , or 4τ , respectively, after the imposition of the sinusoid.

N7.12.2 Capacitors considered to be of high quality may have a dissipation factor smaller than 0.001 at 1 kHz.

N7.12.3 Inductors considered to be of high quality may have a storage factor larger than unity at 200 Hz, where losses are primarily due to wire resistance. At high frequencies, energy dissipation is primarily due to eddy currents, and there is additional loss due to distributed capacitance. An inductor considered to be of high quality may have a storage factor larger than unity at 1 MHz.

N7.13 The accuracy achievable by the three-step procedure depends principally on the ability to determine $(dy/dt)_p$. If $y(t)$ is in the form of a continuous electrical signal, analog differentiation with careful attention to minimizing the effects of hum and noise usually is most effective. If the derivative is determined by sampling at discrete intervals and assuming that $\Delta y/\Delta t$ represents dy/dt , then the interval Δt must be chosen carefully: choice of too small an interval leads to excessive random errors; choice of too large an interval leads to excessive errors of nonlinearity.

N7.14 For a completely self-contained system with no external source of power, this condition is impossible because it implies perpetual motion. However, it is a convenient limit of behavior and often can be approached so closely that the simplifying assumption $\zeta = 0$ can be made with negligible error.

N7.15.1 The equation of the response is

$$(N7.15-1a) \quad y = Y_1 + (Y_0 - Y_1)\alpha^{-1}\epsilon^{-\beta t} \sin(\alpha\omega_n t + \psi)$$

where

$$(N7.15-1b) \quad \alpha = \sqrt{|1 - \zeta^2|}; \quad \beta = \zeta\omega_n; \quad \psi = \cos^{-1}\zeta \quad .$$

N7.15.2 The ratio between the initial amplitude difference $\Delta_0 = |Y_1 - Y_0|$ and the first overshoot $\Delta_{1/2}$ has been termed the *damping factor* and has been used by manufacturers of galvanometer-type instruments to describe the damping of such meters. This measure of damping is useful because, in a well-damped meter, the second overshoot, Δ_1 , may be imperceptible.

N7.15.3 The equation of the response is

$$(N7.15-2) \quad y = Y_1 + (Y_0 - Y_1)\epsilon^{-2t/\tau} (1 + 2t/\tau) \quad .$$

The coordinates of the inflection point are given by

$$t_2 = \tau/2; \quad (y_2 - Y_0)/(Y_1 - Y_0) = 1 - 2/\epsilon \quad .$$

The slope at the inflection point is

$$2(Y_1 - Y_0)/(\epsilon\tau)$$

The tangent at the inflection point intersects Y_1 at a time $3\tau/2$.

The time for the excursion to be almost complete is given in the following table:

Percent completion	90	95	98	99
Multiple of T_n	0.62	0.76	0.93	1.05
Multiple of τ	1.9	2.4	2.9	3.3

N7.15.4 The equation of the response is

$$(N7.15-3a) \quad y = Y_1 + (Y_0 - Y_1)\alpha^{-1}\epsilon^{-\beta t} \sinh(2\zeta\alpha t/\tau + \psi)$$

where

$$(N7.15-3b) \quad \alpha = \sqrt{|\zeta^2 - 1|}; \quad \beta = 2\zeta^2/\tau; \quad \psi = \sinh^{-1}\alpha$$

The coordinates of the inflection point (t_2, y_2) are given by

$$(N7.15-4a) \quad t_2 = [\tau/(4\zeta\alpha)] \cdot \ln[(\zeta + \alpha)/(\zeta - \alpha)]$$

$$(N7.15-4b) \quad (y_2 - Y_0)/(Y_1 - Y_0) = 1 - 2\zeta[(\zeta - \alpha)/(\zeta + \alpha)]^{\zeta/(2\alpha)}$$

The slope of the tangent at the inflection point is $(Y_1 - y_2)/\tau$.

N7.15.5 The difference between y as given by Eq. (N7.15-3) and the y given by

$$y \approx (Y_1 - y_2) \left\{ 1 - \exp[(t_2 - t)/\tau] \right\}$$

is less than 8, 2, or 1 percent when $\zeta = 1, 2,$ or $3,$ respectively.

N7.16 If

$$(N7.16-1a) \quad z = (Y_1 - y)/(Y_1 - Y_0)$$

$$(N7.16-1b) \quad \alpha = \sqrt{|\zeta^2 - 1|}; \quad \beta = 2\zeta^2/\tau = \omega_n\zeta = 2\pi\zeta/T_n$$

$$(N7.16-1c) \quad m = \lambda/\tau$$

$$(N7.16-2a) \quad A = 1 - 2\zeta/(\omega_n\lambda) + (\omega_n\lambda)^{-2} \\ = 1 - m^{-1} + (2\zeta m)^{-2}$$

$$(N7.16-2b) \quad B = (\alpha/\zeta)(1 - 4\zeta^2 m)/(1 + 2m - 4\zeta^2 m)$$

then the responses to the exponential forcing function of Eq. (7.16-1) are given by

$$(N7.16-3) \quad Az = (1/A)\epsilon^{-t/\lambda} + (\lambda\omega_n\alpha)^{-1}\epsilon^{-\beta t}\sin(\omega_n\alpha t + \tan^{-1}B) \quad \text{if } \zeta < 1$$

$$(N7.16-4) \quad A^2z = \epsilon^{-t/\lambda} + (4m^2)^{-1}[1 - 4m + (2 - 4m)(t/\tau)]\epsilon^{-2t/\tau} \quad \text{if } \zeta = 1, \tau \neq 2\lambda$$

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$$(N7.16-5) \quad z = (1 + 2t/\tau + 2t^2/\tau^2)\epsilon^{-2t/\tau} \quad \text{if } \zeta = 1, \tau = 2\lambda$$

$$(N7.16-6) \quad Az = (1/A)\epsilon^{-t/\lambda} + (2m\zeta\alpha)^{-1}\epsilon^{-\beta t}\sinh(2\zeta\alpha t/\tau + \tanh^{-1}B) \quad \text{if } \zeta > 1$$

N7.17 The multiples of τ quoted for $\zeta \geq 1$ are conservative. They are limits as $\zeta \rightarrow \infty$. More precisely, if

$$(N7.17-1) \quad \alpha = \sqrt{|\zeta^2 - 1|}; \quad \beta = 2\zeta^2/\tau = \omega_n\zeta = 2\pi\zeta/T_n$$

$$(N7.17-2) \quad B = 2\zeta\alpha/(2\zeta^2 - 1) \quad ;$$

then

$$(N7.17-3) \quad A = (2\zeta\alpha)^{-1}\epsilon^{-\beta t}\sin(\alpha\omega_n t + \tan^{-1}B) \quad \text{if } \zeta < 1$$

$$(N7.17-4) \quad A = \epsilon^{-2t/\tau}(1 + t/\tau) \quad \text{if } \zeta = 1$$

$$(N7.17-5) \quad A = (2\zeta\alpha)^{-1}\epsilon^{-\beta t}\sinh(2\zeta\alpha t/\tau + \tanh^{-1}B) \quad \text{if } \zeta > 1$$

N7.18 The complete response to a quadratic parabola is given by

$$(N7.18-1) \quad y/b = (t-\tau)^2 + C$$

where

$$(N7.18-2) \quad C = \tau^2(1-A) + \tau^2(A/\alpha)\epsilon^{-\beta t}\sin(\alpha\omega_n t + \tan^{-1}B) \quad \text{if } \zeta < 1$$

$$(N7.18-3) \quad C = \tau^2/2 - (\tau^2/2)\epsilon^{-2t/\tau}(3 + 2t/\tau) \quad \text{if } \zeta = 1$$

$$(N7.18-4) \quad C = \tau^2(1 - A) - \tau^2(A/\alpha)\epsilon^{-\beta t}\sinh(2\zeta\alpha t/\tau + \tanh^{-1}B) \quad \text{if } \zeta > 1$$

where

$$\alpha = \sqrt{|\zeta^2 - 1|}; \quad \beta = \zeta\omega_n = 2\zeta^2/\tau = 2\pi\zeta/T_n$$

$$(N7.18-5) \quad A = (2\zeta^2)^{-1} \\ B = (\alpha/\zeta)(4\zeta^2 - 1)/(4\zeta^2 - 3)$$

N7.19 The exact equations for the response to a pulse of amplitude $(Y_1 - Y_0)$ and duration Δt are

A. For $\zeta < 1$,

$$y_m - Y_0 = (Y_1 - Y_0)\epsilon^{-c}(\epsilon^{2b} - 2\epsilon^b\cos a + 1)^{1/2}$$

$$t_m = (\omega_n\alpha)^{-1}\tan^{-1}[\epsilon^b\sin a / (\epsilon^b\cos a - 1)]$$

B. For $\zeta = 1$,

$$y_m - Y_0 = (Y_1 - Y_0)\epsilon^{-c}(\epsilon^b - 1)$$

$$t_m = \Delta t / (1 - \epsilon^{-b})$$

C. For $\zeta > 1$,

$$y_m - Y_0 = (Y_1 - Y_0)\epsilon^{-c}(\epsilon^{2b} - 2\epsilon^b \cosh a + 1)^{1/2}$$

$$t_m = (\omega_n \alpha)^{-1} \tanh^{-1}[\epsilon^b \sinh a / (\epsilon^b \cosh a - 1)]$$

where

$$a = \omega_n \alpha \bullet \Delta t; \quad b = \zeta \omega_n \bullet \Delta t; \quad c = \zeta \omega_n t_m$$

and the impulse I is given by

$$I = (Y_1 - Y_0) \bullet \Delta t$$

N7.20.1 The transient response becomes academic because it depends on the phase angle of the forcing sinusoid at the instant it is imposed on the system. However, an intuition of the time interval required for the transient response to become small is provided by the information in Sec. 7.15 concerning the response to a step change.

N7.20.2 Although Eq. (7.20-3) represents a fictitious mathematical limit, its simplicity makes it useful for describing instruments, like silicon-crystal force or pressure transducers, which have very little damping and are usually used with $r < 1$.

The equation also indicates that, if the condition $r = 1$ is approached closely enough, destructively large amplitudes of oscillation may occur.

N7.20.3 The two values of frequency ratio r , at which $A - 1 = (A_r - 1)/2$, are given by

$$r_{1/2}^2 = 1 - 2\zeta^2 \pm x(3 - 2x + x^2)^{1/2} / (1 + x)$$

where

$$x \equiv 2\zeta\alpha = 1/A_r$$

Equation (7.20-8) yields ζ with a limit of error of 0.003 only when $\zeta < 0.2$. The difference $\Delta r_{1/2}$ between these two values of $r_{1/2}$ is useful in several scientific fields. It may be used to describe the sharpness of tuning of an electrical circuit or of a mechanical or electro-optical oscillator. With a different mathematical definition—the width of the abscissa segment joining two points whose ordinates are one half the height of the peak—the “bandwidth at half maximum” may also be used to describe the resolving power of a mass spectrometer or optical spectrometer at some stated value of abscissa.

N7.20.4 In a polar plot of this type (a Nyquist diagram), the number of quadrants occupied by the plot is equal to the order of the differential equation describing the physical system (Sec. 7.4). The diagram is particularly valuable in describing such systems because they are capable of self-oscillation if, and only if, the point $(1, \pi)$ lies on or inside of the plot.

N7.23 If the amplitude of Y is estimated to be Y_1 and the maximum rate of change of Y that is of interest is \dot{Y}_m , then the highest frequency of interest is

$$f_m = \dot{Y}_m / (2\pi Y_1)$$

Input signals at frequency $4f_m$ and amplitude $Y_1/8$ could add $0.5 \dot{Y}_m$ to the indicated rate of change if the fidelity of instrument response $y(t)$ were the same at both f_m and $4f_m$.

CHAPTER 8. EXPERIMENTAL DETERMINATION OF THE PARAMETERS OF DYNAMICAL SYSTEMS

8.0 Introduction. If a system is believed to approximate a first- or second-order system, the imposition of a known forcing function, for which the solution is known, may serve to reveal the characteristic parameters of the system. Conversely, observation of the nature of the system's response to some simple forcing function may serve to establish the extent to which a system may be considered to be a first-order or second-order one. Application of the same test to a more complex system may serve to identify one or more simple systems contained within the complex one.

The most convenient forms of forcing function for this purpose are the step, impulse, ramp, and sinusoid.

An alternative technique, the unit-deflection method, applicable to mechanical systems using springs, is a combination of analytical and experimental techniques that permits determination of the natural frequency without recourse to dynamic tests.

The symbol list of Sec. 7.0 also applies to this chapter.

8.1 Test by imposition of a step or impulse. A step change may be imposed, for example,

- (a) on a mechanical spring, by deflecting it a fixed amount and then suddenly releasing it ("plucking");
- (b) on a pneumatic system, by pressurizing it to a steady value and then suddenly releasing the pressure ("burst-diaphragm method");
- (c) on a pressure gauge, by use of a shock tube (the step is of short duration, that depends on the length of the tube);
- (d) on an electrical system, by connecting or disconnecting a source of steady voltage or current ("keying");
- (e) on a thermometer bulb, by sudden immersion in a well-stirred bath of different temperature;
- (f) on a radiometer, by turning on or off a source of incident radiation.

An impulse may be imposed, for example,

- (a) on a mechanical system, by the single stroke of a hammer, or by the single bounce of a pellet;
- (b) on a pneumatic system, by the incidence of a shock or detonation wave;
- (c) on an electrical system, by a capacitor discharge with very short time constant (ballistic method);
- (d) on a radiometer, by use of a short-duration, high-intensity flash.

A. If the response is clearly oscillatory and its decrement is constant (i.e., the ratio between successive amplitudes is constant), the system is like the basic second-order system. The value of ζ may be determined from the measured amplitude ratios by use of Eq. (7.15-3) for the logarithmic decrement δ and by

$$(8.1-1) \quad \zeta^2 = \delta^2 / (\delta^2 + 4\pi^2) \quad .$$

The period of the oscillations may be measured and T_n computed as

$$(8.1-2) \quad T_n = \text{period} \times \sqrt{1 - \zeta^2}$$

B. If the response is clearly oscillatory, but the ratio of successive amplitudes becomes closer to unity as the oscillation continues, there is appreciable square law damping. (See Sec. 10.4.) The period of the oscillations may be measured and assumed equal to T_n . The first overshoot $\Delta_{1/2}$ may be measured and an equivalent ζ computed as

$$(8.1-3) \quad \zeta_m \approx (1/\pi) \ln(\Delta_0/\Delta_{1/2}) \quad \text{if} \quad \zeta_m \ll 1 \quad ,$$

where Δ_0 is the magnitude of the initial step. One may conclude that the system will act like a linear system with $\zeta < \zeta_m$, but additional quantitative deductions are not easily possible.

C. If the response is only slightly oscillatory, or if there is only one noticeable overshoot, the value of ζ may be determined by use of Eq. (8.1-1), where

$$(8.1-4) \quad \delta = 2 \ln(\Delta_0/\Delta_{1/2}) \quad .$$

However, the determination of T_n by use of Eq. (8.1-2) may be of inadequate accuracy. This accuracy may be improved if the damping can be temporarily removed or reduced, thereby permitting use of method A above.

Generally, a system, whose response is so slightly oscillatory that its ζ and T_n cannot be measured accurately, can be treated as a critically damped second-order system, with characteristics as described in Sec. 7.15 C.

D. If the response is aperiodic, with nonzero initial slope, and the graph of dimensionless amplitude $z = (y - Y_1)/(Y_0 - Y_1)$ versus time is approximately a straight line on semilog paper, the system is substantially of the first order and its τ may be determined from the average slope of the line in the highest decade of z . If the exponential character of the response is evident, the plot on semilog paper may be omitted and τ determined immediately as the time when $z = 1/e = 0.368$ or as the absolute value of the reciprocal of the initial slope of $z(t)$ (i.e., the time when the tangent to the response curve at $t = 0$, $z = 1$ intersects the ordinate $z = 0$).

E. If the response to a step is aperiodic, but starts at zero slope, or if the response to an impulse is aperiodic after an initial peak has been reached, so that there is a point of inflection in the response curve, the system may be considered to be of the second order and to be overdamped.

Construct the tangent at the point of inflection (the tangent with maximum positive or maximum negative slope). Let t_3 be the time at which this tangent intersects the ordinate $z = 0$. Let t_2 be the abscissa of the point of inflection. Then

$$(8.1-5) \quad \tau = t_3 - t_2 \quad .$$

If, in response to a step, the point of inflection occurs at $z \geq 0.9$ (implying that $\zeta \geq 3$), a simpler procedure is adequate: Determine the time $t_{1/2}$ at which $z = 1/2$. Then

$$(8.1-6) \quad \tau = t_{1/2}/0.7 \quad .$$

This equation is comparable to

$$(8.1-7) \quad \tau = t_{1/2}/\ln 2$$

for a first-order system.

8.2 Test by imposition of a ramp. If a linear change $Y = at$ is applied to the instrument, the time displacement between the forcing function and the asymptotic steady state of the response represents τ . If the applied ramp change is not accurately linear, such procedure will still yield τ if the asymptote is

drawn as a curve parallel to the forcing function, since the time lag in response to a quadratic parabola also approaches τ .

If the approach to the steady state is oscillatory, the difference between the actual indication and the asymptotic steady state may be replotted or measured directly to provide amplitude ratios and frequencies in the same manner as was done for the step change.

8.3 Test by imposition of a sinusoid or of white noise. Particularly for electrical systems, the response of an instrument to an impressed sinusoidal signal is a convenient way to estimate the characteristic parameters of the instrument. Generally, measurements are made of the amplitude ratio $A = y_1/Y_1$ between response amplitude y_1 and impressed amplitude Y_1 , and of the phase angle φ between the two waves. The procedure is usually effective only when it is possible to vary the forcing frequency f over a wide range. This variation may be effected by continuously varying the forcing frequency while maintaining constant amplitude Y_1 or else by subjecting the instrument to "white noise," as may be developed by a reverse-biased diode (Sec. 5.19), wherein all frequencies of interest are generated virtually simultaneously. A spectrum analyzer or its equivalent is then required. If a white-noise generator is used, there must be careful confirmation that the signal amplitude it generates is the same at all frequencies.

Usually, for second-order systems, the sweep through the band of frequencies of interest seeks to determine the frequency f_r at which a maximum value of A is obtained; this value is the amplitude at resonance A_r (Sec. 7.20).

A. If a system is known, in advance, to be of the first order, then the time constant τ may be determined from measurement of either the amplitude ratio A or the phase angle φ when the steady state has been reached. Convenient formulas are

$$(8.3-1) \quad \tau = \sqrt{3}/(2\pi f) \quad \text{when} \quad A = 1/2 \quad \text{or} \quad \varphi = 60^\circ$$

$$(8.3-2a) \quad \tau = 1/(2\pi f) \quad \text{when} \quad A = 1/\sqrt{2} \quad \text{or} \quad \varphi = 45^\circ$$

where f is the frequency at which the designated value of A or φ occurs. Equation (7.12-2) may be used for other amplitude ratios and phase angles.

B. If a plot, on log paper, of amplitude ratio A versus frequency f appears to be asymptotic to a line of slope -1 as f increases, the system is very nearly of the first order and Eq. (8.3-2a) may be applicable. The value of τ is deducible from

$$(8.3-2b) \quad \tau = 1/(2\pi f_c)$$

where f_c , the corner frequency, is the frequency at which the asymptotic line of slope -1 intersects the ordinate $A = 1$.

C. A polar plot of amplitude ratio A versus phase angle φ , like Fig. 7.20.3, serves to identify both ζ and f_n . The extent to which the plot lies in the second negative quadrant is a measure of the relative importance of the second-order features of the system.

D. The frequency f at which the phase angle φ is 90° is also the natural frequency f_n of a second-order system. Only when $\varphi = 90^\circ$ can f_n be determined without knowledge of ζ . For a first-order system, $\varphi \rightarrow 90^\circ$ as $f \rightarrow \infty$.

E. If appreciable resonance ($A_r > 1$) is detected at some frequency f_r , then the value of ζ is deducible from A_r by the formula

$$(8.3-3) \quad (1 - 2\zeta^2)^2 = 1 - 1/A_r^2$$

and the natural frequency f_n is given by

$$(8.3-4) \quad f_n^2 = f_r^2 \sqrt{1 - 1/A_r^2} \quad .$$

F. For slightly-damped systems ($A_r > 2$), the amplitude ratio A_r yields

$$(8.3-5a) \quad \zeta \approx 1/(2A_r)$$

and the "bandwidth at half-maximum" yields ζ by Eq. (7.20-8), where

$$(8.3-5b) \quad \Delta r_{1/2} \approx (f_{1/2} - f_{-1/2})/f_r$$

and $f_{1/2}$ and $f_{-1/2}$ are the values of f at which the amplitude is $(A_r + 1)/2$. Equation (8.3-5a) determines ζ from an amplitude ratio; Eq. (8.3-5b) determines ζ from a frequency-difference measurement, which may require high accuracy when the bandwidth is narrow.

G. If appreciable resonance is detected at several frequencies, then the formulas in E. and F. may be applied to each of these, so that the behavior of the entire physical system is understood, even though the actual arrangement of its components may not be.

H. An alternative approach is possible if the derivative \dot{y} of the instrument indication can be measured, rather than the indication itself. By Sec. 7.20 G, the forcing frequency at which \dot{y} is a maximum (while forcing amplitude Y_1 is held constant) is also the natural frequency of the instrument, regardless of the value of ζ . Furthermore, if both Y_1 and \dot{y} are measured at the resonant frequency, then ζ can also be determined. This technique is particularly useful when dealing with electrical signals that can be differentiated easily.

8.4 Unit deflection method of determining natural frequency. This method is a combination of analytical and experimental techniques. It generally involves (a) the analytical calculation of a constant that is characteristic of a particular instrument design and (b) the experimental measurement of a deflection. However, the characteristic constant could also be determined empirically.

For a particular design of a simple second-order system there generally exists a relation of the form

$$(8.4-1) \quad f_n^2 \delta = \text{a constant}$$

where δ is an experimentally measured deflection and the constant is characteristic of the instrument design. For any mechanical system where δ is a length, the constant has the dimensions of acceleration. The unit deflection method becomes particularly convenient when the constant is proportional to the local acceleration of gravity g . The determination of f_n then involves no dynamic testing and is not dependent on knowledge of the damping.

A. Linear spring-mass system.

(i) If a concentrated mass m is supported by a massless spring of spring constant k (Sec. 6.1), the deflection δ when gravity acts upon the mass is given by

$$(8.4-2) \quad \delta = mg/k$$

Consequently, by Eq. (6.1-2a),

$$(8.4-3) \quad f_n^2 \delta = g/(4\pi^2)$$

The deflection δ is the change in spring deflection after the assembly is turned 90° so that gravity no longer acts to deflect the spring.

The spring may be of any form: helical, cantilever, diaphragm, etc., provided its mass is negligible, but δ must represent the deflection of the spring itself (the deflection measured in the determination of k) and not of any mechanical magnifying device.

If f_n is in hertz and δ is in inches,

$$(8.4-4a) \quad f_n^2 \delta \approx 10$$

If f_n is in hertz and δ is in centimeters,

$$(8.4-4b) \quad f_n^2 \delta \approx 25 \quad .$$

(ii) If there is no concentrated mass ($m = 0$) and the deflection caused by gravity is due entirely to the distributed mass m_s of the spring, then

$$(8.4-5) \quad f_n^2 \delta = \beta g / (4\pi^2)$$

but β now depends on the form of the spring and usually has a value between 5/4 and 5/3 (Footnote 1).

(iii) If the spring has appreciable mass and there is also a mass concentrated at the end of the spring, then β in Eq. (8.4-5) has a value intermediate between unity and its value in case (ii).

B. Rotational spring-mass system. For the system treated in Sec. 6.3, if δ is the angular deflection caused by an angular acceleration of 1 radian per second², and f_n is in hertz, then

$$(8.4-6) \quad f_n^2 \delta = 1/(4\pi^2)$$

This relationship is principally of academic interest, because of the practical difficulty of creating unit angular acceleration.

C. d'Arsonval galvanometer. For a d'Arsonval galvanometer (permanent magnet, moving coil meter), possessing a coil resistance R and a constant sensitivity S , where

$$(8.4-7) \quad S = (\text{angular coil rotation})/(\text{coil current})$$

the comparable relation is

$$(8.4-8) \quad f_n^2 S / \sqrt{R} = \text{a constant} \quad .$$

The constant depends on the size, shape, and material of the coil and on the flux density in the air gap; it is usually determined empirically. The magnitude of this constant is sometimes taken as a *figure of merit* of moving-coil oscillograph design.

¹For a helical spring or bellows, $\beta = 1.50$

For a cantilever of constant cross section, $\beta \approx 1.56$

For a circular diaphragm with simply supported edges, $\beta \approx 1.56$

For a circular diaphragm with clamped edges, $\beta \approx 1.67$



CHAPTER 9. ANALYTIC DETERMINATION OF THE CHARACTERISTICS OF DYNAMICAL SYSTEMS

9.0 Introduction. In Chapter 6, the differential equations describing the behavior of various systems were derived by use of Newton's law for the balance of forces in mechanical systems or Kirchoff's laws for the balance of currents and emf's in electrical systems. Other laws, such as those for the balance of power or energy, can be used. Some will be illustrated here, because, although they may offer little or no advantage for the treatment of simple first- and second-order systems, they become more useful as systems become complex.

Once a differential equation has been derived, simple algebraic manipulation, generally involving transformation of the dependent variable to a dimensionless form, may allow reduction of the entire equation to a simple canonical form whose behavior is already known. The advantage gained is that inspection of the equation and identification of its characteristic constants is sufficient for an understanding of the behavior of the system, without recourse to detailed solution of each particular equation.

The symbol list of Chapter 7 also applies to this chapter.

9.1 Power equation. A balance between power input and power output was used to establish the differential equation for the thermometer bulb (Sec. 6.12). A similar power balance equation can be used for mechanical or electrical systems.

Example 9.1a. For the linear mechanical system, Sec. 6.1, at any displacement x ,

$$\text{work done by external force} = \int_0^x F \cdot dx = Fx$$

$$\text{energy stored in spring} = \int_0^x kx \cdot dx = kx^2/2$$

$$\text{rate of dissipation of energy in oil} = (b\dot{x}) \cdot \dot{x} = b\dot{x}^2$$

$$\text{energy of motion of mass} = m\dot{x}^2/2$$

The power balance requires that

$$\frac{d}{dt}(Fx) = \frac{d}{dt}\left(\frac{kx^2}{2}\right) + b\dot{x}^2 + \frac{d}{dt}\left(\frac{m\dot{x}^2}{2}\right)$$

so that

$$F\dot{x} = kx\dot{x} + b\dot{x}^2 + m\dot{x}\ddot{x}$$

and dividing through by \dot{x} yields Eq. (6.1-1b).

Example 9.1b. For L , C , and R in series, Sec. 6.8,

$$\text{power delivered by external voltage} = Ei = E\dot{q}$$

$$\text{energy stored in inductance} = Li^2/2 = L\dot{q}^2/2$$

$$\text{energy stored in capacitance} = q^2/(2C)$$

$$\text{rate of dissipation of energy in resistance} = Ri^2 = R\dot{q}^2$$

The power balance requires that

$$\frac{d}{dt}\left(\frac{L\dot{q}^2}{2}\right) + \frac{d}{dt}\left(\frac{q^2}{2C}\right) + R\dot{q}^2 = E\dot{q}$$

so that

$$L\dot{q}\ddot{q} + q\dot{q}/C + R\dot{q}^2 = E\dot{q}$$

and dividing through by \dot{q} yields Eq. (6.8-1a).

9.2 Rayleigh's method of determining natural frequency. This method is used to compute the natural frequency f_n of a system in the absence of appreciable damping. It equates the energy of the system at two instances when the energy is particularly easy to calculate, assuming that at both instances the system is oscillating at frequency f_n [Note N9.2].

For a mechanical system, these instances are

- (1) when oscillation amplitude is a maximum and oscillation velocity is zero, so that potential energy is a maximum;
- (2) when oscillation velocity is a maximum and oscillation amplitude is zero, so that kinetic energy is a maximum.

For an electrical circuit, these instances may be

- (1) when oscillation amplitude (of voltage, current, or charge) is a maximum;
- (2) when the same oscillation amplitude is zero.

Example 9.2a. If the linear mechanical system, Sec. 6.1, is oscillating at its natural frequency so that

$$x = x_m \sin(2\pi f_n t) \quad ,$$

the maximum potential energy occurs when $x = x_m$, $\dot{x} = 0$, and is then

$$kx^2/2 = (1/2)x_m^2 k \quad .$$

The maximum kinetic energy occurs when $x = 0$, $\dot{x} = 2\pi f_n x_m$, and is then

$$m\dot{x}^2/2 = (1/2)(2\pi f_n)^2 x_m^2 m \quad .$$

Equating these two energies leads to

$$(2\pi f_n)^2 = k/m$$

in agreement with Eq. (6.1-2a) .

Example 9.2b. If the current in a circuit with inductance L and capacitance C in series is oscillating at frequency f_n so that

$$i = i_m \sin(2\pi f_n t)$$

the energy is stored entirely in the inductor when i is a maximum. The energy is then

$$(1/2)Li_m^2 .$$

The energy is stored entirely in the capacitor when the capacitor's charge (the time integral of i) is a maximum. The energy is then

$$(2C)^{-1}i_m^2 / (2\pi f_n)^2 .$$

Equating these two energies leads to

$$(2\pi f_n)^2 = 1/(LC)$$

in agreement with Eq. (6.8-2b).

9.3 Lagrange's equation. This equation is usable in the most general case when there is appreciable dissipation (damping). It is derived from the forces, energies, and rate of energy dissipation that exist at any instant. Let

U = kinetic energy of the system

V = potential energy of the system

D = rate of dissipation of energy of the system

ΣF_j = sum of external forces acting upon the system

x_i = independent coordinates on which U , V , and D depend.

Then, for each independent coordinate x_i

$$(9.3-1) \quad \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{x}_i} \right) - \frac{\partial U}{\partial x_i} + \frac{\partial V}{\partial x_i} + \frac{1}{2} \frac{\partial D}{\partial \dot{x}_i} = \Sigma F_j .$$

For mechanical systems, Lagrange's equation is an equation of forces. For other systems, like electrical circuits, the quantities x_i and F_j are *generalized coordinates* and *generalized forces*, respectively, chosen so that they satisfy the condition

$$(9.3-2) \quad (\text{generalized coordinate}) \times (\text{generalized force}) = \text{energy} .$$

Then U is a quadratic function of the *generalized velocities* (the time rate of change of the generalized coordinates) and V is a quadratic function of the generalized coordinates. A separate differential equation (9.3-1) is written for each x_i and the simultaneous solution of these equations leads to a description of system behavior. [Note N9.2]

Example 9.3a. In an electrical circuit containing L , C , and R in series (Fig. 6.8), one may take

$$F_1 = E; \quad x_1 = q$$

so that

$$U = L\dot{q}^2/2; \quad V = q^2/(2C); \quad D = \dot{q}^2 R .$$

Equation (6.8-1a) then follows from Eq. (9.3-1).

Example 9.3b. In an electrical circuit containing L , C , and R in parallel (Fig. 6.9), one may take

$$F_1 = I; \quad x_1 = \int E \cdot dt \equiv u$$

so that

$$U = \dot{u}^2 C / 2; \quad V = u^2 / (2L); \quad D = \dot{u}^2 / R .$$

Equation (6.9-1c) follows from application of Eq. (9.3-1) and the fact that

$$u = L \cdot i_L .$$

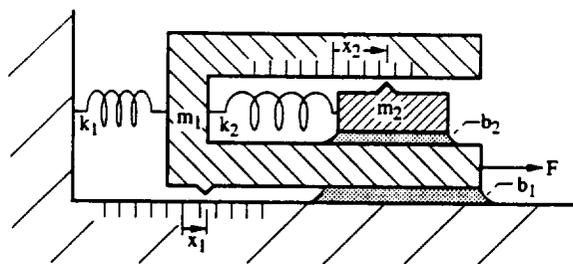


Figure 9.3.—Dynamic vibration absorber.

Example 9.3c. The *dynamic vibration absorber*, Fig. 9.3, uses a smaller damped spring-mass system (m_2 , k_2 , b_2), termed the *absorber*, to reduce the vibration of the almost undamped principal system (m_1 , k_1 , $b_1 \approx 0$) when the latter is subjected to a force F varying sinusoidally at a frequency f that is near to the natural frequency f_{n1} of the principal system.

In the absence of damping ($b_2 = 0$), the otherwise high amplitude of oscillation of system 1, that occurs when $f = f_n$, can be reduced to zero by making the natural frequency f_{n2} of system 2 equal to f_{n1} . However, this reduction would be achieved at the expense of two usually unacceptable consequences:

- (i) the amplitude of oscillation of system 2 is very high, and
- (ii) system 1 has two other resonance points, one on each side of f_{n1} , at which the amplitude of oscillation of system 1 is very high.

By introducing the appropriate amount of damping into system 2 and by selecting an adequately large value of m_2 , a compromise can be achieved in which the amplitudes of oscillation of both systems can be held within acceptable limits over the entire operating frequency range.

In Fig. 9.3, if x_1 and x_2 are the respective elongations of springs 1 and 2, from their respective positions when there is no vibration, and the symbolism parallels that of Secs. 6.1 and 7.0,

$$\begin{aligned} \text{potential energy of system 1} &= (1/2)k_1x_1^2 \\ \text{potential energy of system 2} &= (1/2)k_2x_2^2 \\ \text{kinetic energy of system 1} &= (1/2)m_1\dot{x}_1^2 \\ \text{kinetic energy of system 2} &= (1/2)m_2(\dot{x}_1 + \dot{x}_2)^2 \\ \text{power dissipation in system 1} &= b_1\dot{x}_1^2 \\ \text{power dissipation in system 2} &= b_2\dot{x}_2^2 \end{aligned}$$

Consequently, the dependent variables to be inserted into Eq. (9.3-1) are

$$V = (1/2)k_1x_1^2 + (1/2)k_2x_2^2$$

$$U = (1/2)m_1\dot{x}_1^2 + (1/2)m_2(\dot{x}_1 + \dot{x}_2)^2$$

$$D = b_1\dot{x}_1^2 + b_2\dot{x}_2^2 \quad .$$

If $b_1 = 0$, the following equations result:

$$(1/\omega_{n1}^2)\ddot{y}_1 - (2\zeta a/\omega_{n2})\dot{y}_2 + y_1 - ay_2 = (F/k_1)/x_m$$

$$(1/\omega_{n2}^2)\ddot{y}_2 + (2\zeta/\omega_{n2})\dot{y}_2 + y_2 = - (1/\omega_{n2}^2)\ddot{y}_1$$

where x_m is some reference value of displacement and

$$y_1 = x_1/x_m; \quad y_2 = x_2/x_m; \quad a = k_2/k_1$$

$$\omega_{n1}^2 = k_1/m_1; \quad \omega_{n2}^2 = k_2/m_2; \quad 2\zeta = \omega_{n2}b_2/k_2$$

[Notes N9.2 and N9.3]

NOTES FOR CHAPTER 9

N9.2. Use of f_n , as it has been defined in this chapter, implies that the system is oscillating in its fundamental mode. However, both Rayleigh's and Lagrange's methods are applicable to finding the frequency of higher modes of oscillation (harmonic modes); it is necessary merely to assume the shape of the oscillation or vibration mode and to compute the energies associated with that mode.

N9.3. Optimum conditions for suppression of excessive vibration require a choice of $\beta = m_2/m_1$ and control of ζ to keep maximum vibration-amplitudes sufficiently small. If $F = F_m \sin(2\pi ft)$ and x_m is chosen as F_m/k_1 , then m_2 , b_2 , and k_2 should be chosen so that

$$\omega_{n2}/\omega_{n1} = (1 + \beta)^{-1}$$

$$\zeta^2 = (3/8)\beta/(1 + \beta) \quad .$$

With these choices, the maximum amplitude $x_{1,\max}$ of vibration of system 1 will be given by

$$(x_{1,\max}/x_m)^2 = 1 + 2/\beta$$

and will occur at frequencies given by

$$(f/f_{n1})^2 = \{1 \pm \sqrt{[\beta(2 + \beta)]}/(1 + \beta)\} \quad .$$

The maximum deflection $x_{2,\max}$ of spring 2 will occur at these same frequencies and will be given by

$$x_{2,\max}/x_m \approx 1 + (1/\beta) \quad .$$

An alternate formula for this maximum amplitude, when ζ does not have the optimum value but is near to it, is

$$x_{2,\max}/x_m \approx (0.78/\sqrt{\zeta})(1 + 1/\beta)^{3/4} \quad .$$

The value of β must be large enough so that the amplitude $x_{2,\max}$ is acceptably small.

CHAPTER 10. MORE COMPLEX DYNAMICAL SYSTEMS. ANALOGIES

10.0 Terminology. The symbols of Chapters 6 and 7 will also be used here. Other symbols are defined in Table 10.1 or will be defined as they are introduced.

10.1 Analogies between electrical systems and mechanical or thermal systems. Table 10.1 summarizes results that have been derived in Chapters 6 and 7 concerning typical physical systems obeying the simple first- and second-order linear differential equations with constant coefficients. Not only do the equations for different systems become identical when expressed in terms of Y , y , T , τ , and ζ , but also there appear to exist analogous mathematical relationships for the various dependent physical variables. Thus, any pair of equations may be selected and an analogy drawn between the variables that appear in similar positions in the table. For example, (mass, linear displacement, force) and (moment of inertia, angular displacement, torque) constitute analogous groups of patterns. That these pairings or groupings are not unique is shown by the fact that each of the above groupings is also analogous to both (mass per unit area, particle displacement, pressure) and (mass/area², volume displacement, pressure) for the pneumatic system. Insofar as mechanical and pneumatic systems are concerned, however, there exists a physical analogy as well as a mathematical analogy among the various factors, because all the relations are derivable by use of Newton's second law of motion, and can be converted into each other by simple multiplication by some geometric factor. On the other hand, the analogy between mechanical and electrical systems is solely a mathematical one, rather than a physical one (unless one deals exclusively with power or energy). The analogy between thermal systems and electrical or mechanical systems is also solely of a mathematical nature.

The analogies can be carried over into the treatment of physical systems of greater complexity than those treated in Chapter 6. The analogies may also be used for static problems, where time is not involved, as in problems of the potential (e.g., stress- or strain-contour lines in elasticity¹, current- and voltage-contour lines in an electrically conducting medium, pressure- and velocity-contour lines of hydrodynamic flow, contour lines of heat-flow rate and of temperature in a thermally conducting medium).

In experimentation on mechanical or thermal systems, it is often found that substantial improvements can be achieved in accuracy of measurement, simplicity of apparatus, flexibility of adjustment, or convenience of use, if an analogous electrical system is studied rather than the actual system of interest. This situation is so common that electrical terminology has been carried into other fields (e.g., "acoustic capacitance," "acoustic resistance," "thermal capacitance") and electrical symbols have been used to denote mechanical quantities (e.g., acoustic inertance L_a , mechanical compliance C_m , thermal resistance R_θ). The definitions of such nonelectrical terms are arbitrary and depend upon the particular pair of analogous equations chosen. (The acoustic terms defined in Tables 1-1 and 10.1 are those most popular in the U.S.) Once the analogous pairings have been selected, any type of physical system, however complex, may be represented in analogy.

For dynamical performance studies, it is necessary either that the combinations of physical quantities that have the dimensions of time have the same numerical magnitude in the actual and in the analogous system, or else that they be in constant proportion to each other. (For example, if the constants of an actual second-order system are T and τ and the constants of the analogous second-order system are T' and τ' , then one may choose $T' = nT$ and $\tau' = n\tau$, in which case the "real" time t and the "analog" time t' will be related by $t' = nt$.)

¹Orthogonal quantities are tensile stress and shearing stress, or tensile strain and shearing strain.

In particular, two types of analogies between mechanical and electrical quantities have been used. In one of these analogies, the mechanical quantities (m, b, k, x, \dot{x}, F) are treated as analogous to the electrical quantities ($L, R, 1/C, q, \dot{q}, E$), respectively. This is the more common type of treatment. In another analog arrangement, which is often easier to employ in treating a complex mechanical arrangement, the quantities (m, b, k, x, \dot{x}, F) are treated as analogous to the quantities ($C, G, 1/L, \int edt, e, I$), respectively. Figure 10.1 lists some typical analogous arrangements.

TABLE 10.1.—ANALOGOUS QUANTITIES IN SOME FIRST- AND SECOND-ORDER SYSTEMS.

System	Linear spring-mass (§ 6.1)	Rotational spring-mass (§ 6.3)	U-tube manometer (§ 6.4)	Pneumatic tube and chamber (§ 6.6)
Quantity measured	x (Linear displacement)	ϕ (Angular displacement)	$2x$ (Differential height)	p (Chamber pressure)
Forcing variable	F (Force)	T_e (Torque)	P (Differential pressure)	P (Pressure)
Dependent variable	x (Linear displacement)	ϕ (Angular displacement)	x (Vertical displacement)	$u = Ax$ (Volume displacement)
1st derivative of dependent variable	\dot{x} (Linear velocity)	$\dot{\phi}$ (Angular velocity)	\dot{x} (Vertical velocity)	\dot{u} (Volume velocity)
2nd derivative of dependent variable	\ddot{x} (Linear acceleration)	$\ddot{\phi}$ (Angular acceleration)	\ddot{x} (Vertical acceleration)	\ddot{u} (Volume acceleration)
Coefficient of dependent variable	k (Linear spring constant = $1/\text{compliance}$)	K (Torsional spring constant)	$2\rho g$ ($2 \times$ specific weight)	$\rho a^2/V$ (Acoustic capacitance) ⁻¹
Coefficient of 1st derivative	b (Viscous damping coefficient)	B (Viscous damping coefficient)	$8\eta l/r^2$	$8\pi\eta l/A^2$ (Acoustic resistance)
Coefficient of 2nd derivative	m (Mass)	I (Moment of inertia)	ρl	$\rho l/A$ (Acoustic inductance)
Kinetic energy	$m\dot{x}^2/2$	$I\dot{\phi}^2/2$	$(1/2)\pi r^2 \rho \dot{x}^2$	$\rho l \dot{u}^2 / (2A)$
Potential energy	$kx^2/2$	$K\phi^2/2$	$\pi r^2 \rho g x^2$	$\rho a^2 \dot{u}^2 / (2\rho V)$
Dissipative power	$b\dot{x}^2$	$B\dot{\phi}^2$	$8\pi\eta l \dot{x}^2$	$8\pi\eta l \dot{u}^2 / A^2$
Time constant, τ	b/k	B/K	$4\nu l / (gr^2)$	$8\pi\eta l V / (\gamma p_0 A^2)$
Natural period, T_n	$2\pi\sqrt{m/k}$	$2\pi\sqrt{I/K}$	$\pi\sqrt{2l/g}$	$(2\pi/a)\sqrt{(lV/A)}$
Fraction of critical damping, ζ	$b/\sqrt{4mk}$	$B/\sqrt{4IK}$	$(2\nu/r^2)\sqrt{(2l/g)}$	$(4\pi\nu/a)\sqrt{(lV/A^3)}$
Symbols	A cross-sectional area a speed of sound r tube radius	g local acceleration of gravity l length p pressure	V chamber volume γ specific-heat ratio η viscosity	ν kinematic viscosity ρ density

TABLE 10.1.—Concluded.

System	L, R, C in series (§ 6.8)	L, R, C in parallel (§ 6.9)	L, R in series (§ 6.8)	L, R in parallel (§ 6.9)	C, R in parallel (§ 6.9)	C, R in series (§ 6.8)	Mass moving through a viscous medium (§ 6.10)	Thermometer bulb (§ 6.12)
Quantity measured	e_C (Capacitor voltage)	i_L (Inductor current)	e_R	i_L	i_R	e_C	v (Velocity)	θ (Temperature)
Forcing variable	E (Voltage)	I (Current)	E	I	I	E	F (Force)	Θ (Temperature)
Dependent variable	q (Charge)	$i e \cdot dt$	i	$i e \cdot dt$	e	q	v	θ
1st derivative of dependent variable	$\dot{q} = i$ (Current)	e (Voltage)	di/dt	e	\dot{e}	i	\dot{v}	$\dot{\theta}$
2nd derivative of dependent variable	\ddot{q}	\ddot{e}						
Coefficient of dependent variable	$1/C$ (Elastance = $1/\text{Capacitance}$)	$1/L$ (= $1/\text{Inductance}$)	R	$1/L$	$1/R$	$1/C$	b (Force/velocity)	Conduction kA/l (Thermal conductance)
Coefficient of 1st derivative	R (Resistance)	$G = 1/R$ (Conductance)	L	$1/R$	C	R	m (Mass)	Convection hA (Thermal conductance)
Coefficient of 2nd derivative	L (Inductance)	C (Capacitance)						Radiation $4\sigma cA\theta^3$ (Thermal conductance)
Conserved energy	$(L^2/2) + q^2/(2C)$	$(L^2/2) + (C^2/2)$	$L^2/2$	$L^2/2$	$C^2/2$	$q^2/(2C)$	$mv^2/2$	Conduction mc (Thermal capacitance)
Dissipated power	i^2R	e^2/R	i^2R	e^2/R	e^2/R	i^2R	bv^2	Convection mc (Thermal capacitance)
Time constant, τ	RC	L/R	L/R	L/R	RC	RC	m/b	Conduction mc/l (kA)
Natural period, T_n	$2\pi\sqrt{LC}$	$2\pi\sqrt{LC}$						Radiation $mc/(4\sigma cA\theta^3)$
Fraction of critical damping, ξ	$(R/2)\sqrt{C/L}$	$[1/(2R)]\sqrt{L/C}$						
Symbols	A area c specific heat	k thermal conductivity h heat transfer coefficient	l length m mass	ϵ emittance σ Stefan-Boltzmann constant				

	Linear mechanical system	Pneumatic system	"Series" electrical system	"Parallel" electrical system	Thermal system
<p>Basic Laws: (1) Sum of forces (including inertial reaction) acting at a point = 0. (2) $\Sigma F = m\ddot{x}$</p>	<p>Basic Laws: (1) Sum of potential drops around a loop = 0. (2) $E = L (di/dt)$</p>	<p>Basic Laws: (1) Sum of currents at a point = 0. (2) $q = CE$</p>	<p>Basic Laws: (1) Sum of temperature drops around a loop = 0. (2) $\Delta H = mc\Delta\theta$</p>		
<p>Linear displacement, x Linear velocity, \dot{x} Mass, m Linear compliance, $1/k$ Linear mechanical resistance, b Force, F</p>	<p>Volume displacement, V Volume velocity, \dot{V} Acoustic inertance, L_a Acoustic capacitance, C_a Acoustic resistance, R_a Pressure difference, P</p>	<p>Charge, q Current, i Inductance, L Capacitance, C Resistance, R Voltage, E</p>	<p>Heat, H Power, \dot{H}</p> <p>Thermal capacitance, C_θ Thermal resistance, R_θ Temperature, θ</p>		
<p>Elements</p>	<p>Elements</p>	<p>Elements</p>	<p>Elements</p>	<p>Elements</p>	<p>Elements</p>
<p>1</p>	<p>2</p>	<p>1</p>	<p>1</p>	<p>1</p>	<p>None</p>
<p>2</p>	<p>2</p>	<p>2</p>	<p>2</p>	<p>2</p>	<p>C_θ</p>
<p>3</p>	<p>3</p>	<p>3</p>	<p>3</p>	<p>3</p>	<p>R_θ</p>

Figure 10.1. --Mathematically analogous physical systems.

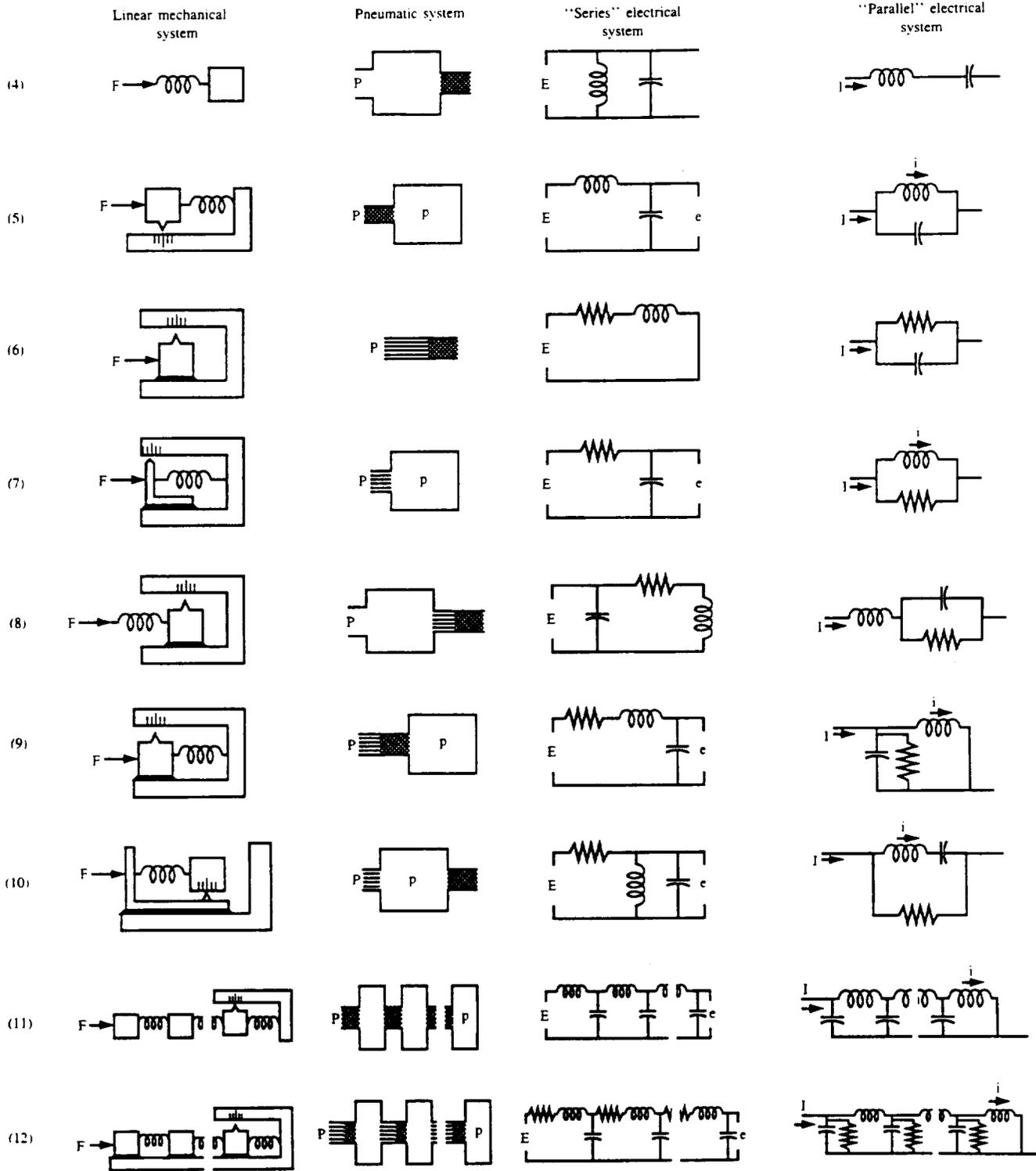


Figure 10.1.—Continued.

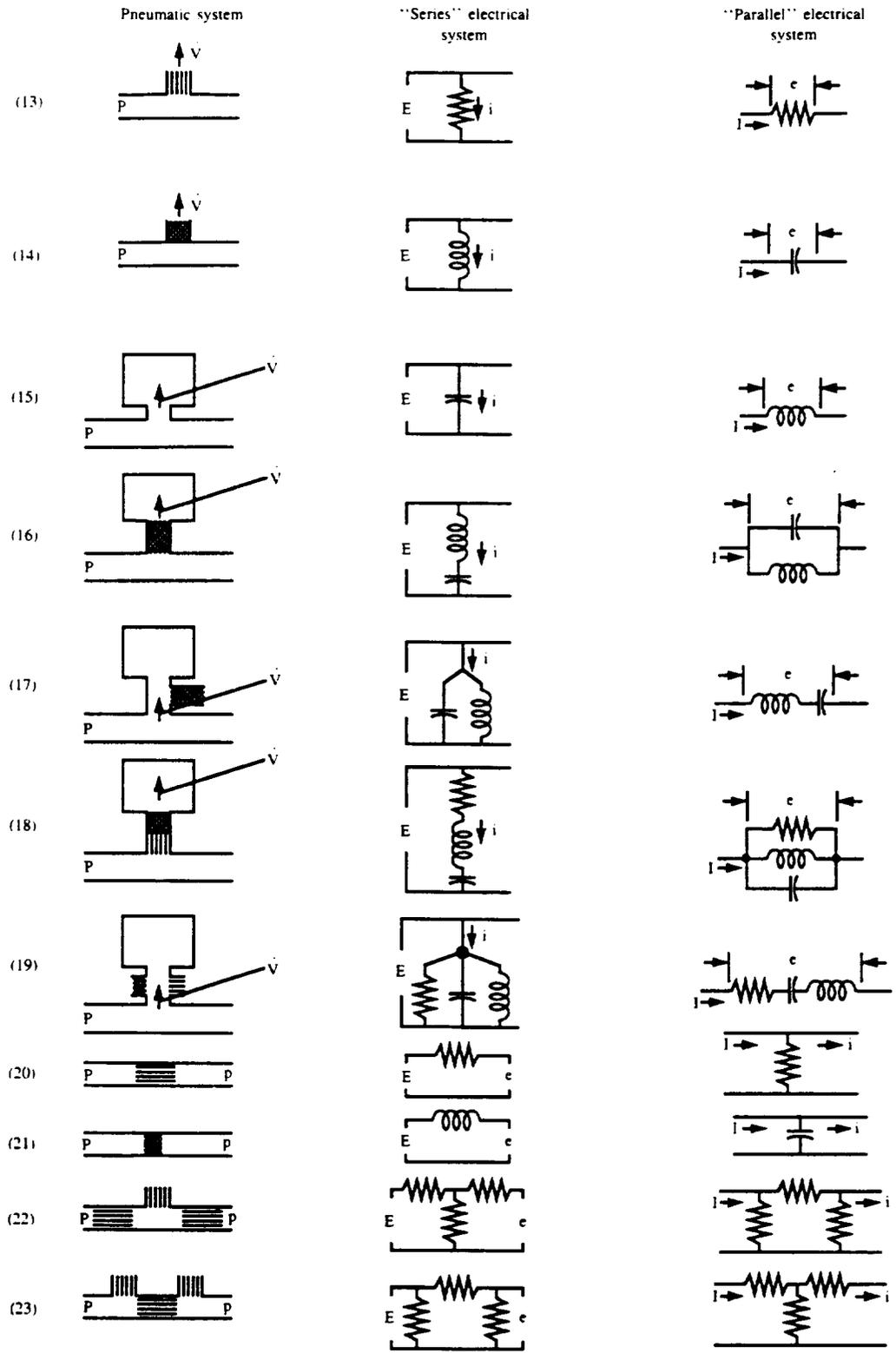


Figure 10.1—Concluded.

10.2 Complex systems. The equations of the two simple types of systems treated in detail in Chapter 7 are describable mathematically as first- and second-order linear differential equations with constant coefficients. From the mathematical viewpoint, more complex systems may involve equations of the same order, but with variable coefficients or with nonlinear terms, or they may involve equations of higher order. From the physical viewpoint, these complex systems may involve situations where the parameters T , ζ , or τ are themselves functions of time (a case of variable coefficients), where the damping is proportional to a higher power of the generalized velocity (Sec. 9.3) than the first power (a case of nonlinearity), or where there is more than one simple second-order system and the several systems are linked together (yielding a higher-order system). An extreme but common example of the last situation is the case where some or all of the constants are distributed rather than lumped.

In the case of first- or second-order systems with variable coefficients or nonlinear terms, it is sometimes adequate to treat the variable terms as though they were constant over a limited region of operation or to replace the nonlinear terms by linear ones over a limited region of operation, and thereby to utilize the results that have already been obtained in Chapter 7. Such approximations generally represent a considerable savings in effort over the treatment of the exact equations.

In the case of higher-order systems which are linear, it is possible, as indicated in Sec. 7.4, to represent these as a tandem combination of component subsystems that are of the first or second order. Separate differential equations

$$(10.2-1) \quad f_i(y_i, \dot{y}_i, \ddot{y}_i) = 0$$

may be written for each of these subsystems. Each of the equations (10.2-1) has its own τ or its own T_n , depending on whether f_i represents a first- or second-order equation, respectively. Another algebraic equation or parametric set of equations may be needed to describe the relation among the dependent variables

$$(10.2-2) \quad F(y_1, y_2, \dots, y_i, \dots) = 0$$

if such a relation is not already included in Eq. (10.2-1). The simultaneous solution of all of these equations will lead to an exact description of the entire system.

An example of a system in which this procedure is needed is the dynamic vibration absorber (Example 9.3c) [Note N10.2]. On the other hand, mere inspection of the individual equations (10.2-1) may provide an intuitive perception of the system's behavior.

There are also many occasions where an adequate indication of system behavior can be obtained by use of approximation techniques that treat the complex system as though it were a simple first- or second-order system. Conditions that are sufficient to permit such approximations and to yield adequately accurate deduction of the forcing function $Y(t)$ from the response $y(t)$ are

- A. in a serial arrangement of subsystems represented by Eq. (10.2-1), the energy used by any subsystem is only a small fraction of the energy available from the preceding subsystem;
- B. sufficient time has elapsed for the decay of the transients that occurred on the initiation of $Y(t)$;
- C. the value ω_{\max} of the highest angular frequency in $Y(t)$ that is of interest is small compared to the values of $1/\tau$ and ω_n of the subsystems. This condition may be excessively conservative (although sufficient, it may not be necessary), as will be indicated in Example 10.5a.

In this chapter, some examples of approximation techniques will be presented.

10.3 First-order system with variable time constant. If τ is not constant, but depends on either t or y so that

$$(10.3-1) \quad \tau = \tau(t) \quad \text{or} \quad \tau = \tau(y)$$

then the forcing function $Y(t)$ may be deduced from the response $y(t)$ by piecewise application of the procedure outlined in Sec. 7.13. Each point (t,y) is replaced by (t,Y) , where

$$(10.3-2) \quad Y = y + \tau(dy/dt)$$

and τ is given by Eq. (10.3-1). The practical limitation of this procedure is usually determined by the accuracy of determining dy/dt .

An approximation to this procedure is possible if Λ_{\min}^* , as defined in Sec. 7.13, is much larger than the largest value of τ . Then, each point (t,y) may be replaced by $(t + \tau, y)$. The resultant curve is $Y(t)$.

10.4 Second-order system with square-law damping. In some instruments such as air-damped accelerometers, iron-vane-type ammeters, and pneumatic systems containing orifices, a considerable part of the damping force is proportional to the square of the velocity rather than to the first power of the velocity. For such and similar systems, the equation of dynamic response may be written as

$$(10.4-1) \quad \ddot{y}/(4\pi^2f_n^2) + b_2|\dot{y}| + b_1\dot{y} + y = Y$$

in place of Eq. (7.1-2a).

The behavior of this system can be understood by writing the response equation as

$$(10.4-2a) \quad \ddot{y}/(4\pi^2f_n^2) + \tau\dot{y} + y = Y$$

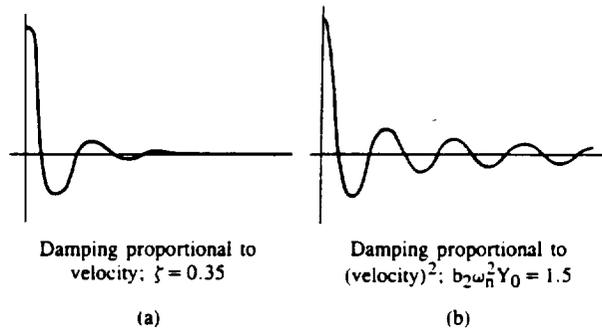
where

$$(10.4-2b) \quad \tau = b_2|\dot{y}| + b_1 = \zeta/(\pi f_n).$$

The system resembles that of Sec. 7.1, except that the time constant τ and the fraction of critical damping ζ vary with the velocity amplitude $|\dot{y}|$. At sufficiently large values of $|\dot{y}|$, where $\zeta \gg 1$, the system may appear overdamped; at sufficiently small values of $|\dot{y}|$, if $b_1 \ll 1/(\pi f_n)$, the system may appear to have negligible damping.

A system with predominantly square-law damping that appears to be well damped for large velocity amplitudes of the forcing function will appear almost undamped for very small velocity amplitudes of the forcing function. Only the first-power (linear) damping that invariably remains in any real physical system then serves to continue to dissipate energy.

The comparative response of a linear system and a square-law system (when $b_1 \ll b_2|\dot{y}|$) to a step change that produces the same first overshoot is shown in Fig. 10.4. In response to a step change of magnitude



(a) System with linear damping.
(b) System with square-law damping.

Figure 10.4.—Response to a step change.

($-Y_0$), from an initial value of Y_0 , the approach to the steady-state value of 0 will be through a damped oscillation. The ratio between successive amplitudes of the oscillation will not be constant, as in the basic linear system, but will approach unity as the oscillation continues. Thus, the oscillation will be attenuated less and less as the oscillation continues. The approach to the final value would never become aperiodic, no matter how great b_2 may be, if b_1 were zero. [Note N10.4]

If the system is overdamped because b_1 in Eq. (10.4-2b) is appreciable, then the system may be considered to resemble a first-order system with slightly variable τ , and the techniques of Sec. 10.3 may be applicable.

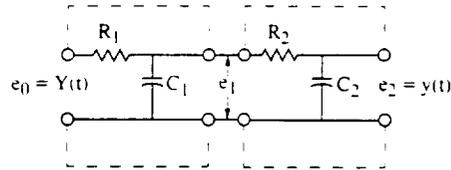


Figure 10.5.1—Two first-order systems in tandem.

10.5 Examples of higher-order systems in which superposition is possible.

Example 10.5a. The circuit of Fig. 10.5.1 may represent

- (1) an electrical circuit consisting of two first-order systems connected in tandem, under the restriction that $C_2 \ll C_1$
- (2) the analog of a resistance-thermometer winding centered in a protective well that is immersed in a flowing fluid whose temperature is to be measured. The analogous quantities are

e_0	the temperature of the fluid
R_1	the thermal resistance of the fluid in contact with the well
C_1	the thermal capacitance of the well
e_1	the temperature of the well
R_2	the thermal resistance of the material separating the resistance-thermometer winding from its protective well
C_2	the thermal capacitance of the resistance-thermometer winding
e_2	the temperature of the winding

These quantities are defined more precisely in Table 10.1 and Sec. 6.12.

If condition A of Sec. 10.2 exists, the output $e_1(t)$ of the first system may be determined from the input $Y(t)$, the time constant τ_1 , and the information in Secs. 7.7 to 7.13, neglecting the presence of the second system. This output may then be treated as the input to the second system alone.

The tandem combination of the two systems acts like an overdamped second-order system with time constant $\tau_1 + \tau_2$.

Example 10.5b. The circuit of Fig. 10.5.2 may represent

- (1) an electrical circuit consisting of a first-order system connected to a second-order system (Sec. 6.8), under the restriction that $C_2 \ll C_1$
- (2) the analog of a thermocouple measuring fluid temperature, as in Sec. 6.12, generating an electrical output that is measured by an electromagnetic oscillograph, as in Sec. 6.3. The

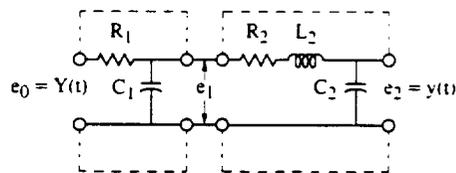


Figure 10.5.2—A first-order and a second-order system in tandem.

analogous quantities are

- e_0 the temperature of the fluid
- R_1 the thermal resistance of the fluid in contact with the thermocouple
- C_1 the heat capacity of the thermocouple
- e_1 in turn,
 - (i) the temperature of the thermocouple
 - (ii) the electric current delivered to the oscillograph because of the emf generated by the thermocouple
- R_2 the electromagnetic damping torque of the oscillograph produced by eddy currents and Lenz law currents
- L_2 the rotational inertia of the oscillograph
- C_2 the spring stiffness of the oscillograph
- e_2 the actual oscillograph deflection

These quantities are defined more precisely in Table 10.1 and Secs. 6.3 and 6.12.

- (3) the analog of a pressure gauge, as in Sec. 6.1, being used to measure the pressure in the chamber of a pneumatic system, as in Sec. 6.6, whose capillary is so small or so long that the pneumatic system is overdamped. The analogous quantities are

- e_0 pressure at the entrance of the capillary
- R_1 the pneumatic resistance of the capillary
- C_1 the compressibility of the gas in the chamber
- e_1 the pressure in the chamber
- R_2 the mechanical damping of the pressure gauge (often negligible)
- L_2 the mechanical inertia of the pressure gauge
- C_2 the spring stiffness of the pressure gauge
- e_2 the pressure indicated by the gauge

These quantities are defined more precisely in Table 10.1 and Secs. 6.6 and 6.7.

The pressure gauge volume may be included as part of the chamber volume entering into C_1 . If a correction $\alpha\gamma p_0$ for the change in volume V (Case E of Sec. 6.7) is needed, the condition

$$\alpha\gamma p_0 \ll V$$

should be met.

The output of the first-order system may be determined from the input $Y(t)$, the time constant τ_1 , and the information in Secs. 7.7 to 7.13, neglecting the presence of the second-order system. This output may then be treated as the input to the second-order system alone, using the information in Secs. 7.15 to 7.21.

If $\tau_2 \ll \tau_1$ and $T_{n2} \ll \tau_1$ then $e_2(t)$ closely resembles $e_1(t)$.

If $\zeta_2 > 1$, then

$$e_2(t) \approx Y(t - \tau_1 - \tau_2).$$

Example 10.5c. The diagram of Fig. 10.5.3 may be considered to represent

- (1) an electrical circuit consisting of two second-order systems, connected in tandem, with $C_2 \ll C_1$;

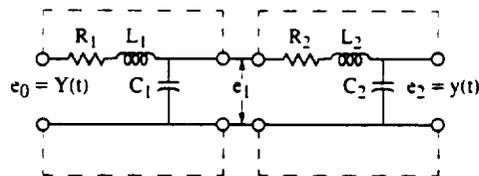


Figure 10.5.3—Two second-order systems in tandem.

- (2) a pressure gauge, as in Sec. 6.1, connected to the chamber of a pneumatic system, as in Secs. 6.6 and 6.7. The analogous quantities are the same as in Example 10.5b(3) and, in addition,

L_1 the mass of gas in the pneumatic tube.

If $C_2 \ll C_1$ and if $Y(t)$ contains no sinusoidal component of appreciable amplitude that has a frequency comparable to f_{n2} , the output of the first system may be determined from $Y(t)$, the parameters τ_1 and f_{n1} , and the information in Secs. 7.15 to 7.21, neglecting the presence of the second system. This output may then be treated as the input to the second system alone.

If $\tau_2 \ll \tau_1$ and $T_{n2} \ll T_{n1}$, then $e_2(t)$ closely resembles $e_1(t)$. Fidelity in achieving the condition

$$e_2(t) \approx Y(t)$$

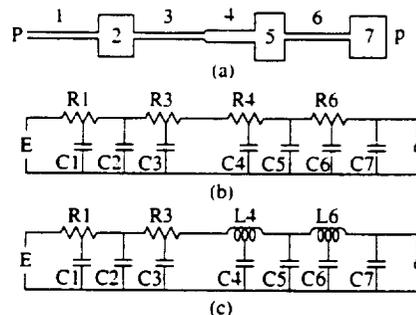
is always more likely if ζ_1 and ζ_2 are close to 0.7.

If $\zeta_1 \gg 1$, $\zeta_2 \gg 1$, then

$$e_2(t) \approx Y(t - \tau_1 - \tau_2).$$

Example 10.5d. Figure 10.5.4(a) shows a pneumatic system consisting of an arbitrary arrangement of capillary tubes and volumes. Its approximation depends on whether the imposed pressure changes are sufficiently slow so that fluid inertia effects can be neglected. For any one of the tubes, a reasonable criterion is that f shall be less than $\pi\nu/A$, where

- f frequency of the imposed pressure fluctuations
- ν kinematic viscosity of the gas
- A cross-sectional area of the tube



(a) Pneumatic system.
 (b) "Series" electrical analog when inertial effects are absent.
 (c) "Series" electrical analog when inertial effects are present.

Figure 10.5.4—Several systems in tandem.

A. If this criterion is satisfied for every one of the tubes, the system consists solely of acoustic resistances and capacitances, and is analogous to the circuit of Fig 10.5.4(b). The entire system can be treated as a first-order one with an equivalent time constant that is a function of the separate time constants of various portions of the system. The data of Table 10.1 and Sec. 6.7 may be used to compute these time constants. For the basic combination of a tube and a chamber, the time constant may be written as

$$\tau = BC(V + Al/2)$$

where

$$B = 8\nu/(\pi a^2); \quad C = \pi^2 l/A^2.$$

Here B is a property of the gas, C is a property of tube geometry, V is chamber volume, and $Al/2$ represents one half of the tube volume.

For the specific circuit of Fig. 10.5.4(b), the equivalent time constant of the entire system, representing the response e to the forcing function E is

$$\tau = R_1 \left(\frac{C_1}{2} + \sum_2^7 C_i \right) + R_3 \left(\frac{C_3}{2} + \sum_4^7 C_i \right) + R_4 \left(\frac{C_4}{2} + \sum_5^7 C_i \right) + R_6 \left(\frac{C_6}{2} + C_7 \right) .$$

Each summation of capacitances in this formula is analogous to the summation of all volumes downstream of the respective tube.

B. If the criterion ($f < \pi\nu/A$) is not satisfied for some particular tube, the inertia of the gas in the tube may become significant. For example, if the criterion is not satisfied for tubes 4 and 6 in Fig. 10.5.4(a), the electrical analogy becomes Fig. 10.5.4(c). Usually, this system may be treated conveniently as the superposition of one first-order system and two second-order systems only when criteria A, B, C of Sec. 10.2 are met. If they are not met, and the subsystems are underdamped, the system's response to a step or impulse may show oscillations at each of the resonant frequencies of the subsystems and at the sums and differences of those frequencies. Conversely, if the transient response to a step or impulse does show such oscillations, it is unlikely that superposition techniques will be useful in estimating the system's behavior. Solution using Eqs. (10.2-1) and (10.2-2) may be necessary, unless the system is of the types treated in Sec. 10.6 [Note N10.5].

10.6 Systems with uniformly distributed parameters. When parameters like resistance, inductance, and capacitance, or their analogs, are uniformly distributed along a length l of considerable magnitude, the system's behavior often resembles that of a first-order or second-order system. Four examples will be given here.

- (1) A line of considerable length that behaves predominantly like a first-order system
- (2) A line of considerable length that is terminated by a "load" in such manner that the system behaves like a first-order system
- (3) A line of considerable length that behaves in most respects like a second-order system
- (4) A line of considerable length that is terminated by a "load" in such manner that the system behaves like a second-order system

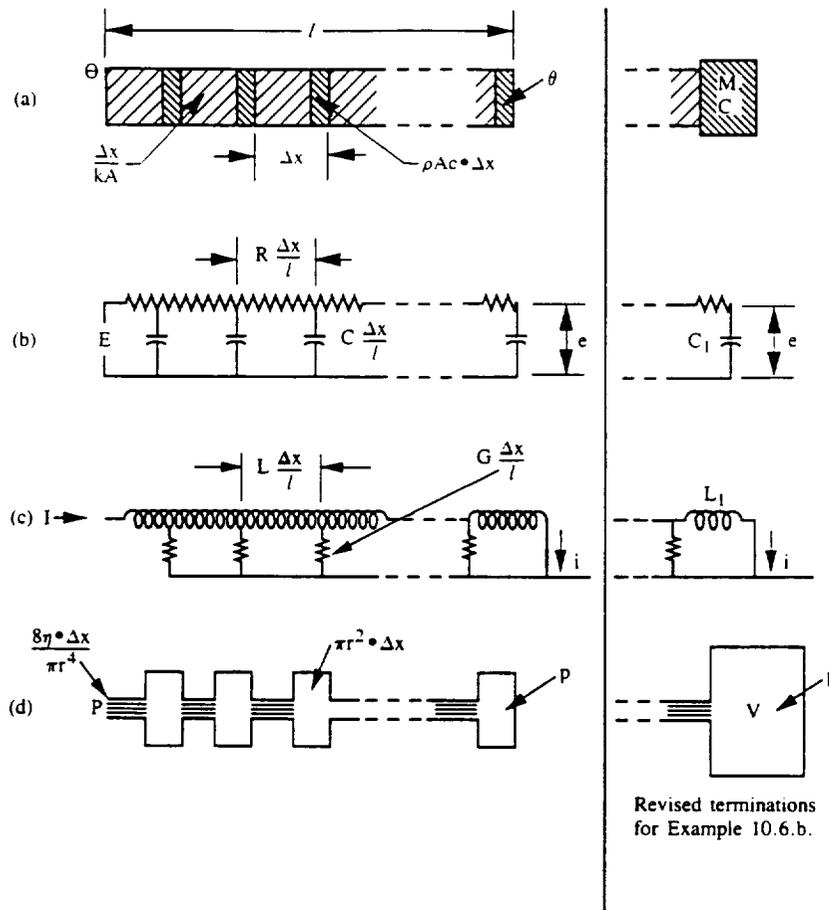
Example 10.6a. The problem of temperature change of one end of a long bar of constant cross section (Fig. 10.6.1(a)) whose other end is subjected to a fluctuating temperature Θ , where the bar has length l , cross-sectional area A , density ρ , thermal capacity (specific heat) c , and thermal conductivity k , may be approached by dividing the length of the bar into elements of length Δx , each of which is considered as a first-order system similar to that of Sec. 6.12. Each element has mass $\rho A \Delta x$, thermal capacity c , and thermal resistance $\Delta x / (kA)$.

The analogous "series" electrical circuit may be represented by two parallel conducting wires, each of length l and resistance $R/2$, such that the capacitance between them is C . This system may be represented by a sequence of first-order systems whose elements are the resistance $(R/l) \Delta x$ and the capacitance $(C/l) \Delta x$. This system is shown in Fig. 10.6.1(b).

The analogous "parallel" electrical system may be represented by Fig. 10.6.1(c), wherein the elements are $(L/l) \Delta x$ and $(G/l) \Delta x$.

An analogous pneumatic system is a section of a long gas-filled tube of uniform cross section. If inertia forces are negligible compared to viscous forces, any element of length Δx has an acoustic resistance $(8\pi\eta/A^2) \Delta x$ and an acoustic capacitance $A \Delta x / (\gamma p_0)$. The equivalent model is shown in Fig. 10.6.1(d).

The output (temperature in Fig. 10.6.1(a), open-circuit emf in Fig. 10.6.1(b), short-circuit current in Fig. 10.6.1(c), and pressure in Fig. 10.6.1(d)), measured at one end of the system, in response



Revised terminations for Example 10.6.b.

- (a) Thermal system.
- (b) "Series" electrical system.
- (c) "Parallel" electrical system.
- (d) Pneumatic system.

Figure 10.6.1—Long, uniform line represented as a first-order system.

to an input at the other end, resembles that of a first-order system (Secs. 7.7 to 7.13) having a time constant τ equal to

$$\begin{aligned}
 & l^2 / (2\kappa) && \text{for the thermal system } (\kappa \equiv \text{diffusivity}) \\
 & RC/2 && \text{for the "series" electrical system} \\
 (10.6-1) \quad & LG/2 && \text{for the "parallel" electrical system} \\
 & 4\pi vl^2 / (a^2 A) && \text{for the pneumatic system [Note N10.6.1]}
 \end{aligned}$$

provided that condition C of Sec. 10.2 is met. If it is not, then the system behaves like an overdamped second-order system (Secs. 7.15 to 7.21), with τ as given by (10.6-1).

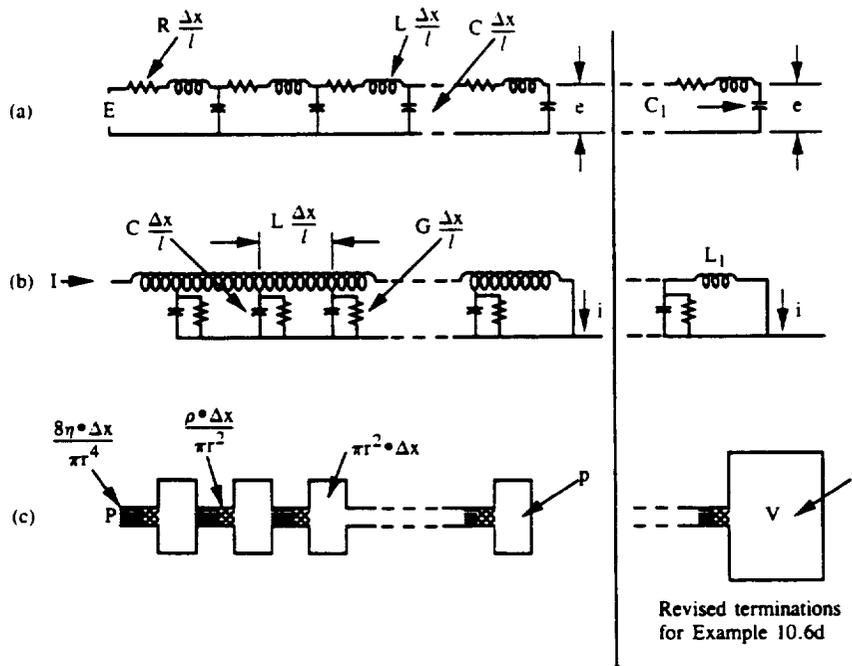
Example 10.6b. If Example 10.6a were modified by adding a termination at the output end as shown at the right-hand side of Fig. 10.6.1 (a larger thermal mass, a larger capacitor, a larger inductor, or a larger volume), then the systems would behave as though their time constants were

$$\begin{aligned}
 & \frac{l}{kA} \left(\frac{\rho A l c}{2} + MC \right) && \text{for the thermal system} \\
 (10.6-2) \quad & R \left(\frac{C}{2} + C_1 \right) && \text{for the "series" electrical system} \\
 & G \left(\frac{L}{2} + L_1 \right) && \text{for the "parallel" electrical system} \\
 & \frac{8\pi\nu l}{A^2 a^2} \left(\frac{Al}{2} + V \right) && \text{for the pneumatic system}
 \end{aligned}$$

The behavior would be similar to that described for Example 10.6 a, except that, as any termination became a larger fraction of the quantity shown in parentheses, the system's behavior would approach more nearly that of a basic first-order system.

Example 10.6c. If, in the electrical and pneumatic systems like those of Example 10.6a, the distributed inductance of the pneumatic system, the distributed inductance of the series electrical system, and the distributed capacitance of the parallel electrical system cannot be neglected, the models of these systems must be replaced by those of Fig. 10.6.2. Each of these systems possesses the following features:

- (1) It is capable of oscillation. In particular, it is also capable of oscillation at frequencies that are harmonics of its fundamental mode.



(a) "Series" electrical system.
 (b) "Parallel" electrical system.
 (c) Pneumatic system.

Figure 10.6.2.—Long, uniform line represented as a second-order system.

- (2) In its fundamental mode, it behaves like a second-order system having a natural period T_n and time constant τ given in Table 10.6c:

Table 10.6c

System	T_n^2	τ
Pneumatic tube [Note N10.6.2]	$16 l^2/a^2$	$4\pi v l^2/(Aa^2)$
"Series" electrical line	$16 LC$	$RC/2$
"Parallel" electrical line	$16 LC$	$LG/2$

- (3) In response to a suddenly applied input, no response will be obtained at the output until a *delay time* $\tau_0 \approx T_n/4$ has elapsed. Thereafter, the system's response will resemble that of the second-order system whose natural period and time constant are listed above.

Example 10.6d. If Example 10.6c were modified by adding a termination at the output end as shown at the right-hand side of Fig. 10.6.2 (a larger volume, capacitor, or inductor), then the systems would retain the characteristics listed for Example 10.6c except that

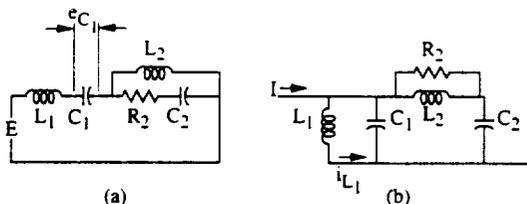
- (a) the natural periods and time constants listed in Table 10.6c would be multiplied by the factors listed in Table 10.6d
- (b) the delay time τ_0 would still be approximately equal to $T_n/4$, where T_n is listed in Table 10.6c.

Table 10.6d.—Multipliers for T_n^2 and τ Listed in Table 10.6c.

System	Multiplier for T_n^2	Multiplier for τ
Pneumatic [Note N10.6.2]	$1 + \pi V/(4l^2)$	$1 + 2V/(\pi r^2 l)$
"Series" electrical	$1 + \pi^2 C_1/(4C)$	$1 + 2C_1/C$
"Parallel" electrical	$1 + \pi^2 L_1/(4L)$	$1 + 2L_1/L$

NOTES FOR CHAPTER 10

N10.2 The electrical analogues of Fig. 9.3 are shown in Fig. 10.2. A convenient method of obtaining the steady-state solutions for such circuits is indicated in Note N10.5. The differential equations of Example 9.3c are derivable in such manner.



(a) "Series" circuit analog.
 (b) "Parallel" circuit analog.

Figure 10.2.—Electrical analogs of the dynamic vibration absorber (Fig. 9.3).

§Ch.10-Notes

N10.4 In the extreme case where b_1 is negligible, the system is characterized by two constants, like b_2 and f_n , and a third constant that has the dimensions of y . Let

$$\beta = 4\pi^2 f_n^2 b_2 Y_0 .$$

If y_n is the ordinate of the n th overshoot ($n = 0, 1, 2, \dots$), and ordinates with n even are of opposite sign to ordinates with n odd, then the successive ordinates are given by

$$(1 - 2\beta z_{n+1}) \exp(2\beta z_{n+1}) = (1 + 2\beta z_n) \exp(-2\beta z_n)$$

where

$$z_n = |y_n|/Y_0$$

and

$$z_0 = 1 .$$

An empirical approximation is

$$z_n = (1 + 1.4n\beta)^{-1}$$

except for the case ($n = 1, \beta > 1$).

For the first overshoot ($n = 1$),

$$z_1 \approx 1 - 4\beta(1 - \beta)/3 \quad \text{if} \quad \beta \leq 0.3$$

$$z_1 \approx 1/(2\beta) \quad \text{if} \quad \beta \geq 3$$

The time interval between the initiation of the step and the first crossing of the asymptote is approximately

$$(1 + \beta/4)/(4f_n) \quad \text{if} \quad \beta < 10$$

$$0.2\beta^{2/3}/f_n \quad \text{if} \quad \beta > 10 .$$

Succeeding oscillations occur at a frequency approximately equal to f_n .

N10.5 A convenient way to determine the steady-state response of any of the electrical analogs shown in this chapter, to a sinusoidal forcing function of angular frequency ω , is to represent the impedances of inductances and capacitances as $j\omega L$ and $1/(j\omega C)$, respectively (where $j = \sqrt{-1}$) and then to determine the ratio (output/input) for the particular circuit. Thereby, the problem becomes an algebraic one, rather than one of differential equations.

N10.6.1 This result is valid when Poiseuille-type flow exists; namely, that, if a pressure fluctuation

$$P = P_m \sin(\omega t)$$

occurs at the inlet end of the tube, the condition

$$A\omega/\nu < 2\pi^2$$

is then satisfied. Here, A is the cross-sectional area of the capillary tube and ν is the kinematic viscosity of the fluid.

N10.6.2 The pneumatic time constant given in Table 10.6c is correct only when there is Poiseuille-type flow. The criterion for this condition is given in Note N10.6.1. When Poiseuille-type flow does not occur, inertial forces become dominant over viscous forces, the system is almost undamped, and τ may be neglected.

The resultant natural vibrations ("organ-pipe vibrations") may obscure a measurement, made at the outlet end of the tube, that is intended to measure the instantaneous value of a fluctuating inlet pressure. An expedient used to provide a reliable measurement is to make the tube of substantially infinite length and to measure the pressure at some convenient location at a relatively short distance x from the inlet end. The installation of a pressure gauge at this location must be made in such manner that

- (a) there is no significant change in the cross-sectional area of the gas flow path;
- (b) there are no protrusions in the gas flow path;
- (c) any cavity between the gas flow path and the pressure gauge is of negligible acoustic capacitance or inductance.

If a pressure fluctuation

$$P = P_m \sin(\omega t)$$

occurs at the inlet end of a circular tube of radius r , the pressure at a distance x from the inlet end will be attenuated so that the amplitude is

$$P_m e^{-\alpha x}$$

where

$$\alpha = \beta (ar)^{-1} \sqrt{\omega \nu}$$

provided the tube is of substantially infinite length, so that reflections from the output end will be so weak that organ-pipe oscillations will not interfere appreciably with the measurement. When $A\omega$ is so small that Poiseuille-type flow exists ($A\omega < 2\pi^2\nu$), the proportionality constant β is equal to 2. At larger values of $A\omega$, where inertial forces become prominent, the proportionality constant β , for air, is near to unity.

"Substantially infinite length" implies a length l such that $e^{-\alpha l} < 1$. For example, if $\alpha l > 3$, the attenuation at the outlet end will be more than 20-fold.

This technique of pressure measurement is termed the "infinite-line technique." In acoustics, it forms the basis of some designs of probe microphones.

In the analogous electrical or thermal systems, the infinite line technique remains applicable. The electrical analogies have often been used, in either experimental or mathematical form, to analyze pneumatic situations. In these analogs, the attenuation constant α is given by

$$\alpha = \sqrt{\omega \tau / l^2}$$

where τ is given in (10.6-1).

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Reference Tables

- 1-1. Physical quantities in Mechanics
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- 1-5. Some quantities in Radiation and Illumination
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TABLE 1-1.—PHYSICAL QUANTITIES IN MECHANICS

Symbol, definition	Name	System of dimensions	
		M,L,T	F,L,T
l	Length	L	L
t	Time	T	T
m	Mass	M	$FL^{-1}T^2$
F	Force	MLT^{-2}	F
ϕ	Angle	1	1
Ω	Solid angle	1	1
A	Area	L^2	L^2
V	Volume	L^3	L^3
v	Velocity, linear	LT^{-1}	LT^{-1}
c_a	Speed of sound	LT^{-1}	LT^{-1}
$\dot{\phi}$	Velocity, angular	T^{-1}	T^{-1}
\dot{V}	Velocity, volume	L^3T^{-1}	L^3T^{-1}
\dot{m}	Velocity, mass	MT^{-1}	$FL^{-1}T$
$a = \ddot{v}$	Acceleration, linear	LT^{-2}	LT^{-2}
g	Acceleration of gravity, local	LT^{-2}	LT^{-2}
g_n	Acceleration of gravity, standard	LT^{-2}	LT^{-2}
$\alpha = \ddot{\phi}$	Acceleration, angular	T^{-2}	T^{-2}
$G_0 = F l^2 / (m_1 m_2)$	Gravitational constant	$M^{-1}L^3T^{-2}$	$F^{-1}L^4T^{-4}$
$J_A = A l^2$	Moment of area	L^4	L^4
$I_m = m l^2$	Moment of inertia	ML^2	FLT^{-2}
$\rho = m/V$	Density	ML^{-3}	$FL^{-4}T^2$
$1/\rho$	Specific volume	$M^{-1}L^3$	$F^{-1}L^4T^{-2}$
$w = \rho g_n$	Specific weight	$ML^{-2}T^{-2}$	FL^{-3}
mg	Weight	MLT^{-2}	F
$P_m = mv$	Momentum, linear	MLT^{-1}	FT
$M = Fl$	Moment of a force	ML^2T^{-2}	FL
$I_\phi = I_m \dot{\phi}$	Moment of momentum (angular momentum)	ML^2T^{-1}	FLT
$i = Ft$	Impulse	MLT^{-1}	FT
$s = Wt$	Action	ML^2T^{-1}	FLT
$p = F/A$	Pressure	$ML^{-1}T^{-2}$	FL^{-2}
$P = W/t$	Power (Flux)	ML^2T^{-3}	FLT^{-1}
P/A	Power density (flux density)	ML^{-3}	$FL^{-1}T^{-1}$
$\Delta P / \Delta f$	Power spectral density	ML^2T^{-2}	FL
$W = Fl$	Energy	ML^2T^{-2}	FL
W/V	Energy density	$ML^{-1}T^{-2}$	FL^{-2}
$\mu_f = F_1/F_2$	Coefficient of friction, sliding	1	1
$\mu_\phi = lF_1/F_2$	Coefficient of friction, rolling	L	L
$\Gamma = F/l$	Surface tension coefficient	MT^{-2}	FL^{-1}
$\eta = Fl/(Av)$	Viscosity	$ML^{-1}T^{-1}$	$FL^{-2}T$
$1/\eta$	Fluidity	$M^{-1}LT$	$F^{-1}L^2T^{-1}$
$\nu = \eta/\rho$	Kinematic viscosity	L^2T^{-1}	L^2T^{-1}
$D = \dot{m}/(A \cdot \Delta\rho/\Delta l)$	Diffusion coefficient	L^2T^{-1}	L^2T^{-1}

TABLE 1-1.—Concluded.

Symbol, definition	Name	System of dimensions	
		M,L,T	F,L,T
Al_1l_2	Product of inertia of an area	L^4	L^4
ml_1l_2	Product of inertia of a body	ML^2	FL^{-1}
$Z = I_A/l$	Section modulus	L^3	L^3
$\epsilon_n = \Delta l/l$	Strain, normal (longitudinal)	1	1
$\gamma = \Delta\phi$	Strain, shear	1	1
$\sigma_n = F/A$	Stress, normal	$ML^{-1}T^{-2}$	FL^{-2}
$\tau = F/A$	Stress, shear	$ML^{-1}T^{-2}$	FL^{-2}
$\beta = (\Delta V/V)/\Delta p$	Compressibility	$M^{-1}LT^2$	$F^{-1}L^2$
$1/\beta$	Bulk modulus	$ML^{-1}T^{-2}$	FL^{-2}
$E_n = \sigma_n/\epsilon_n$	Modulus of tension (modulus of elasticity, Young's modulus)	$ML^{-1}T^{-2}$	FL^{-2}
$G = \tau/\gamma$	Modulus of shear	$ML^{-1}T^{-2}$	FL^{-2}
$C_a = V\beta \propto \Delta V/\Delta p$	Acoustic compliance	$M^{-1}L^4T^2$	$F^{-1}L^5$
$R_a = \Delta p/\Delta \dot{V}$	Acoustic resistance	$ML^{-4}T^{-1}$	$FL^{-5}T$
$L_a = m/A^2$	Acoustic inertance	ML^{-4}	$FL^{-5}T^2$
\dot{V}	Acoustic source strength	L^3T^{-1}	L^3T^{-1}
$\Delta c_a/\Delta f$	Acoustic dispersion	L^2T^{-1}	L^2T^{-1}
$C_S = \Delta p/\Delta v = \rho c_a$	Specific acoustic resistance	$ML^{-2}T^{-1}$	$FL^{-3}T$

TABLE 1-2.—PHYSICAL QUANTITIES IN HEAT

Symbol, definition	Name	Dimensions	
		MLT θ	WLT θ
θ	Temperature	θ	θ
m	Mass	M	$WL^{-2}T^2$
$\rho = m/V$	Density	ML^{-3}	$WL^{-5}T^2$
W	Energy	ML^2T^{-2}	W
P	Power	ML^2T^{-3}	WT^{-1}
$C_\theta = \Delta W/\Delta\theta$	Thermal capacity	$ML^2T^{-2}\theta^{-1}$	$W\theta^{-1}$
$c = \Delta W/(m \cdot \Delta\theta)$	Specific heat	$L^2T^{-2}\theta^{-1}$	$L^2T^{-2}\theta^{-1}$
$\Delta W/(V \cdot \Delta\theta)$	Thermal capacity per unit volume	$ML^{-1}T^{-2}\theta^{-1}$	$WL^{-3}\theta^{-1}$
$\Delta W/\Delta m$	Latent heat	L^2T^{-2}	L^2T^{-2}
$H = W/m$	Enthalpy	L^2T^{-2}	L^2T^{-2}
$\Delta S = \Delta W/\theta$	Entropy	$ML^2T^{-2}\theta^{-1}$	$W\theta^{-1}$
$\Delta S/m$	Specific entropy	$L^2T^{-2}\theta^{-1}$	$L^2T^{-2}\theta^{-1}$
$h = P/(A \cdot \Delta\theta)$	Heat transfer coefficient	$MT^{-3}\theta^{-1}$	$WL^{-2}T^{-1}\theta^{-1}$
$k_\lambda = P/(A \cdot \Delta\theta/\Delta l)$	Thermal conductivity	$MLT^{-3}\theta^{-1}$	$WL^{-1}T^{-1}\theta^{-1}$
$\kappa = k_\lambda/(\rho c)$	Thermal diffusivity	L^2T^{-1}	L^2T^{-1}
$\alpha_L = \Delta l/(l \cdot \Delta\theta)$	Coefficient of expansion, linear	θ^{-1}	θ^{-1}
$\alpha_V = \Delta V/(V \cdot \Delta\theta)$	Coefficient of expansion, cubical	θ^{-1}	θ^{-1}
$k = W/\theta$	Boltzmann constant	$ML^2T^{-2}\theta^{-1}$	$W\theta^{-1}$
$R_0 = W/(m\theta)$	Universal gas constant	$L^2T^{-2}\theta^{-1}$	$L^2T^{-2}\theta^{-1}$

*Applicable laws: $\Delta W = \Delta(pV) = \Delta(\theta S)$

TABLE 1-3.—ANALOGOUS QUANTITIES IN ELECTRICITY AND MAGNETISM

Electricity			Magnetism		
Symbol, definition	Name	Dimensions, RILT system		Name	Symbol, definition
I	Current	I		Magnetic pole strength	m^*
$Q = It$	Charge	IT	RIT	Magnetic flux	$\Phi = \beta m^*$
$\Psi = \beta Q$	Electric flux	IT	RIT	Magnetic moment	$M_m = m^* l$
$M_q = Ql$	Electric moment	ILT	RILT	Magnetic induction	$B = \Phi/A$
$D = \Psi/A$	Electric displacement	$IL^{-2}T$	$RIL^{-2}T$	(magnetic flux density)	$= \mu_0 H + \beta J$
$= \epsilon_0 \mathcal{E} + \beta P$				Magnetic polarization	$J = M_m/V$
$P_q = M_q/V$	Electric polarization	$IL^{-2}T$	$RIL^{-2}T$	(magnetization, intensity of magnetization)	$= \mu_0 \chi_m H$
$= \epsilon_0 \chi_q \mathcal{E}$				Magnetomotive force	$\mathcal{F} = \Delta W/m^*$
$E = \Delta W/Q = IR$	Electromotive force (electric potential difference)	RI	I	(magnetic potential difference)	
$\mathcal{E} = E/l = F/Q$	Electric field strength (electric field intensity, electric potential gradient)	RIL^{-1}	IL^{-1}	Magnetic field strength (magnetic field intensity, magnetizing force)	$H = \mathcal{F}/l = F/m^*$
$\epsilon = D/\mathcal{E}$	Permittivity (absolute dielectric constant)	$R^{-1}L^{-1}T$	$RL^{-1}T$	Permeability	$\mu = B/H$
ϵ_0	Permittivity of a vacuum	$R^{-1}L^{-1}T$	$RL^{-1}T$	Permeability of a vacuum	μ_0
$\epsilon_r = \epsilon/\epsilon_0$	Relative permittivity (dielectric constant, specific inductive capacity)	1	1	Relative permeability	$\mu_r = \mu/\mu_0$
$\chi_{q,v} = (\epsilon_r - 1)/\beta$	Electric susceptibility per unit volume	1	1	Magnetic susceptibility per unit volume	$\chi_{m,v} = (\mu_r - 1)/\beta$
Applicable laws					
$F = \beta Q_1 Q_2 / (4\pi \epsilon l^2)$	Coulomb's law			Coulomb's law $F = \beta m_1^* m_2^* / (4\pi \mu l^2)$	
$\mathcal{E} = -\beta (d\Phi/dt)/l$	Faraday's law			Faraday's law $E = L(di/dt)$	

TABLE 1-4.—ADDITIONAL PHYSICAL QUANTITIES
IN ELECTRICITY AND MAGNETISM

Symbol, definition	Name	Dimensions, RILT system
t	Time	T
l	Length	L
A	Area	L ²
V	Volume	L ³
m	Mass	RI ² L ⁻² T ³
F	Force	RI ² L ⁻¹ T
P	Power	I ² R
W	Energy	I ² RT
R	Resistance	R
$g_q = 1/R$	Conductance	R ⁻¹
$\rho_q = RA/l$	Resistivity	RL
$\rho_q^{-1} = 1/(AR)$	Conductivity	R ⁻¹ L ⁻¹
$C = Q/E$	Capacitance	R ⁻¹ T
L	Inductance	RT
$\mathcal{R} = \mathcal{F}/\Phi_m$	Reluctance	R ⁻¹ T ⁻¹
$\Lambda = 1/\mathcal{R}$	Permeance	RT
$\alpha_q = M_q/(\epsilon_0 \mathcal{E})$	Polarizability	L ³
$d_{mn} = Q/F$	Piezoelectric coefficient	R ⁻¹ I ⁻¹ L ⁻¹
$S_\theta = \Delta E/\Delta\theta$	Seebeck coefficient (Thermoelectric power)	RI Θ
$\mu_T = \Delta E/\Delta\theta$	Thomson coefficient	RI Θ^{-1}
$\Pi = P/I$	Peltier coefficient	RI
$R_H = \Delta E/(BI/l)$	Hall coefficient	I ⁻¹ L ³ T ⁻¹
$P_E = \Delta\theta/(BI/l)$	Ettinghausen coefficient	R ⁻¹ I ⁻² L ³ T ⁻¹ Θ
$Q_N = \Delta E/(B \cdot \Delta\theta)$	Nernst coefficient	L ² T ⁻¹ Θ^{-1}
$S_L = (d\theta/db)/(B \cdot d\theta/dl)$	Righi-Leduc coefficient	R ⁻¹ I ⁻¹ L ² T ⁻¹

TABLE 1-5.—SOME QUANTITIES IN RADIATION AND ILLUMINATION

Radiation		Dimen- sions, PLT system	Illumination	
Symbol, definition	Name		Name	Symbol, definition
Q_e	Radiant energy	PT	Luminous energy	Q_v
$w_e = Q_e/V$	Radiant energy density	PTL ⁻³	Luminous energy density	$w_v = Q_v/V$
$\dot{\Phi}_e = \dot{Q}_e$	Radiant flux	P	Luminous flux	$\dot{\Phi}_v = \dot{Q}_v$
$I_e = \dot{\Phi}_e/\Omega$	Radiant intensity of source	P	Luminous intensity of source	$I_v = \dot{\Phi}_v/\Omega$
$M_e = \dot{\Phi}_e/A$	Radiant exitance of source (radiant flux density)	PL ⁻²	Luminous exitance of source (luminous flux density)	$M_v = \dot{\Phi}_v/A$
$E_e = \dot{\Phi}_e/A$	Irradiance on receiver	PL ⁻²	Illuminance on receiver	$E_v = \dot{\Phi}_v/A$
$L_e = \dot{\Phi}_e/(A\Omega)$	Radiance of source or through cross section of path	PL ⁻²	Luminance of source or through cross section of path (was brightness)	$L_v = \dot{\Phi}_v/(A\Omega)$
		1	Luminous efficacy (luminosity)	$K_v = \dot{\Phi}_v/\dot{\Phi}_e$
		1	Luminous efficiency (relative luminosity)	$K_\lambda = K_{v,\lambda}/K_{v,max}$
		1	Maximum luminous efficacy	$K_{v,max}$
μ'	Absorption coefficient	L ⁻¹		
μ'/ρ	Mass absorption coefficient	L ⁴ P ⁻¹ T ⁻³		
$n = c_{vac}/c$	Refractive index	1		
$\frac{2(n_1 - n_2)}{(n_1 + n_2 - 2)}$	Dispersive power	1		
φ/l	Rotatory power	L ⁻¹		
$\alpha = \varphi/(lp)$	Specific rotatory power (Specific rotation)	L ⁴ P ⁻¹ T ⁻³		

Common quantities		
Symbol	Definition	Dimensions
l	Length	L
A	Area	L ²
V	Volume	L ³
φ	Angle	1
Ω	Solid angle	1
c	Speed of light	LT ⁻¹
m	Mass	PL ² T ⁻³
ρ	Density	PL ⁻¹ T ⁻³

TABLE 1-5.—Concluded.

Subscripts:	
e	physical variable, related to radiation (sometimes omitted if illumination is not being considered)
v	psychophysical variable, related to illumination
v,max	the value at peak of standard human visibility curve
λ	(may be a second subscript) the value, at wavelength λ , of the derivative of the quantity with respect to λ . (The derivative has different dimensions.)
Suffixes:	
i)	Terms with the ending "ivity" denote the intrinsic properties of material with a clean, polished surface.
ii)	Terms with the ending "ance" denote the extrinsic properties of an actual specimen of a specified shape and surface condition.
Some dimensionless quantities: For any specified type of radiation—	
ρ (reflectance)	or ρ' (reflectivity) = Φ_r/Φ_i
ϵ (emittance)	or ϵ' (emissivity) = Φ_e/Φ_{BB}
α (absorptance)	or α' (absorptivity) = $\Phi_a/\Phi_i = \Phi_a/\Phi_{BB}$
τ (transmittance)	= Φ_t/Φ_i
R (reflectance factor)	= Φ_r/Φ_D
where	
Φ_i = incident flux	Φ_a = absorbed flux Φ_t = transmitted flux
Φ_r = reflected flux	Φ_e = emitted flux
Φ_{BB} = flux that would be emitted or absorbed by a blackbody under the same conditions*	
Φ_D = flux that would be reflected by a perfect diffuser under the same conditions*	
* "Same conditions" implies same specimen temperature, same radiation wavelengths, and same geometric conditions (such as specimen area, angles of incidence and view, and solid angles of the incident and reflected beams).	
Applicable laws:	
$\rho_\lambda + \alpha_\lambda + \tau_\lambda = 1$	
$\alpha_\lambda = \epsilon_\lambda = 1 - \rho_\lambda$ for an opaque body	
Note: For more complete information, see Dictionary of Lighting Terminology, Illuminating Engineering Society (IES) Lighting Handbook, 1981.	

TABLE 1-6.—SOME DIMENSIONLESS NUMBERS

Name	Subscript of <i>N</i>	Definition	Name	Subscript of <i>N</i>	Definition
Biot	<i>Bi</i>	hl/k	Mach	<i>M</i>	v/c_a
Clausius	<i>Cl</i>	$v^3\rho/(k \cdot \Delta\theta)$	Nusselt	<i>Nu</i>	hl/k
Eötvös	<i>Eo</i>	$\rho l^2 g/\Gamma$	Peclet	<i>Pe</i>	vl/κ
Euler	<i>Eu</i>	$p/(\rho v^2)$	Poisson	<i>Po</i>	$(E_n/2G) - 1$
Fanning ^a	<i>Fa</i>	$\tau/(\rho v^2/2)$	Prandtl	<i>Pr</i>	$c\eta/k$
Fourier	<i>Fo</i>	$\kappa/(\omega l^2)$	Rayleigh	<i>Ra</i>	$g\beta l^3 \cdot \Delta\theta/(\nu\kappa)$
Froude ^b	<i>Fr</i>	v^2/gl	Reynolds	<i>Re</i>	vl/ν
Graetz	<i>Gz</i>	$mc/(kl)$	Schmidt	<i>Sc</i>	ν/D
Grashof	<i>Gr</i>	$g\beta l^3 \cdot \Delta\theta/\nu^2$	Stanton	<i>St</i>	$h/(\rho cv)$
Knudsen	<i>Kn</i>	λ/l	Stokes	<i>Sk</i>	$\nu\nu/(gl^2)$
Lewis	<i>Le</i>	D/κ	Strouhal	<i>Sr</i>	$\omega l/v$
Symbols					
<i>c</i>	specific heat at constant pressure	<i>v</i>	velocity		
<i>c_a</i>	speed of sound	β	coefficient of cubical expansion		
<i>D</i>	diffusion coefficient	Γ	surface tension coefficient		
<i>E_n</i>	modulus of tension	η	viscosity		
<i>G</i>	modulus of shear	$\Delta\theta$	temperature difference		
<i>g</i>	local acceleration of gravity	κ	thermal diffusivity		
<i>h</i>	heat transfer coefficient	λ	mean free path of molecules		
<i>k</i>	thermal conductivity	ν	kinematic viscosity		
<i>l</i>	characteristic length	ρ	density		
\dot{m}	mass flow rate	τ	shear stress		
<i>p</i>	pressure	ω	angular frequency		

^aFanning factor $f = N_{Fa}(r_h/l)$, where r_h = hydraulic radius, l = length over which Δp occurs, $\Delta p = \tau \times (l/r_h)$.
^bAlternatively, the square root of the expression listed.

TABLE 2-1.—LIST OF EQUAL AMOUNTS

Notes:

1. The symbol in parentheses following the name of the unit is the accepted symbol for the unit, if such symbol exists.
2. An equality sign represents an exact relation established by some international standards organization; an approximation sign (\approx) indicates that the equality is correct only to the number of significant figures given.
3. The word "mile" alone denotes a U.S. statute mile. The word "second" alone denotes an International Atomic Time (TAI) second.

4. *Numerics:*

$$c_a = 0.980\ 665$$

$$c_m = 0.453\ 592\ 37$$

$$c_0 = 2.997\ 924\ 58 \times 10^{10}$$

$$c_p = 1.01325$$

5. *Abbreviations:*

esu	electrostatic system unit
emu	electromagnetic system unit
Gu	Gaussian system unit
HLu	Heaviside-Lorentz system unit
mksu	unrationalized meter-kilogram-second unit
Slu	Système International unit

6. Abbreviations for units should be written without a period and in the singular.

7. The American National Standards Institute (ANSI) recommends that a central dot be used to separate the symbols for two units appearing in a product.

Geometric Quantities

Length:

The meter is the distance travelled by an electromagnetic wave in a vacuum in $(100/c_0)$ second.

$$1 \text{ meter (m)} = 1 \times 10^{10} \text{ \AA} = 10^6 \text{ micron } (\mu)$$

$$= 10^{15} \text{ fermi} = (1/0.0254) \text{ inch (in)} = (1/0.3048) \text{ foot (ft)}$$

$$= (1/1852) \text{ U.S. nautical mile.}$$

[Superseded names: micron = $1 \mu\text{m}$; fermi = 1 fm]

$$1 \text{ X-unit} \approx 1.002 \times 10^{-13} \text{ m}$$

$$1 \text{ foot (ft)} = 12 \text{ inch (in)} = (1/3) \text{ yard (yd)}$$

$$= (1/5280) \text{ mile (mi)}$$

$$1 \text{ mile} = (463/402.336) \text{ U.S. nautical mile}$$

$$= (33/38) \text{ British nautical mile}$$

$$1 \text{ astronomical unit (AU) (mean distance between Sun and Earth)}$$

$$\approx 149.6 \times 10^9 \text{ m}$$

$$1 \text{ light year} \approx 9.4605 \times 10^9 \text{ m}$$

$$1 \text{ parsec} \approx 308.6 \times 10^{10} \text{ m}$$

$$\approx 3.26 \text{ light year}$$

$$\approx 2.06 \times 10^5 \text{ AU}$$

$$1 \text{ kayser} = 1 \text{ cm}^{-1} = \text{a unit of wave number} = (\text{wave length})^{-1}$$

Area:

$$1 \text{ square meter (m}^2) = 10^{-4} \text{ hectare (ha)} = 10^{28} \text{ barn}$$

$$1 \text{ acre} = 4840 \text{ square yard}$$

$$1 \text{ circular mil} = (\pi/4) \times 10^{-6} \text{ in}^2$$

Volume:

$$1 \text{ cubic meter (m}^3) = 1000 \text{ liter} = 10^9 \text{ lambda } (\lambda)$$

$$= 1 \text{ stere (st)} \approx 264.172 \text{ gallon}$$

$$1 \text{ gallon} = 231 \text{ in}^3$$

$$1 \text{ barrel of oil (bbl)} = 42 \text{ gallon}$$

Plane angle:

$$1 \text{ radian} = (180/\pi) \text{ degree} = (200/\pi) \text{ grade (grad)}$$

Solid angle:

$$1 \text{ steradian (sr)} = (2/\pi) \text{ spherical right angle}$$

$$= (180/\pi) \text{ square degree}$$

$$1 \text{ sphere} = 4\pi \text{ steradian} = 8 \text{ spherical right angles}$$

Geometric Quantities and Time

Time and Frequency:

$$1 \text{ second (s)} = 1 \text{ ephemeris second} = (1/86400) \text{ mean solar day}$$

$$\approx 1.002\ 737\ 91 \text{ sidereal second}$$

$$1 \text{ hertz (Hz)} = 1 \text{ cycle per second (cps), in reference only to a continuous, periodic phenomenon}$$

$$1 \text{ baud} = 1 \text{ pulse per second}$$

$$1 \text{ becquerel (Bq)} = 1 \text{ Slu of activity} = 1/\text{second}$$

$$= 1 \text{ Slu of radioactive source strength} = (10^{-10}/3.7) \text{ curie (Ci)}$$

Speed:

$$1 \text{ meter per second} = (1/0.3048) \text{ foot per second (fps)}$$

$$= (1/0.44704) \text{ mile per hour (mph)}$$

$$1 \text{ knot (kn)} = 1 \text{ U.S. nautical mile per hour}$$

$$\approx 1.150\ 779 \text{ mile per hour (mph)}$$

$$1 \text{ cubic meter per second} = 1000 \text{ liter per second}$$

$$= 2118.88 \text{ cubic foot per minute (cfm)}$$

$$= 127132.8 \text{ cubic foot per hour (cfh)}$$

$$1 \text{ U.S. gallon per minute (gpm)} = (231/1728) \text{ cfm}$$

$$\approx 0.003\ 785\ 41 \text{ cubic meter per minute}$$

$$\text{speed of light in vacuo} = (c_0/100) \text{ m/s}$$

Acceleration:

$$1 \text{ m/s}^2 = 100 \text{ Gal} = 10^{11} \text{ Eötvös} = 1/(10c_a) \text{ "g"} \approx 3.280\ 840 \text{ ft/s}^2$$

$$\text{standard acceleration of gravity } (g_n) = 1 \text{ "g"} = 1 \text{ "g unit"}$$

$$= 10 c_a \text{ m/s}^2 = (10 c_a/0.3048) \text{ ft/s}^2$$

$$\approx 32.174\ 049 \text{ ft/s}^2$$

Mass:

1 kilogram mass (kgm or kg) = 1 Slu of mass
 = 0.001 tonne = 0.001 metric ton = 5000 carat
 = 10^9 gamma (γ) = $1/c_m$ pound mass (lbm or lb)
 = $[0.03048/(c_p c_m)]$ slug = $(7000/c_m)$ grain
 $\approx 6.0222 \times 10^{26}$ atomic mass unit (amu)

1 slug = 1 geepound
 $\approx 32.174\ 049$ pound mass (lbm or lb)

1 pound mass = 7000 grain

Force:

1 newton (N) = 1 Slu of force = 10^5 dyne
 = 0.001 sthene (sn) = $(100/c_p)$ gram force (gf)
 = $[1/(0.3048\ c_m)]$ poundal (pdl)
 = $[0.1/(c_p c_m)]$ pound force (lbf) $\approx 0.224\ 809$ lbf

1 pound force (lbf) = $1000\ c_m$ gram force (gf)
 $\approx 32.174\ 049$ pdl

Weight:

This term has been used to mean either
 "(mass) \times (standard acceleration of gravity)," or
 "(mass) \times (local acceleration of gravity)."

The units of weight are the units of force.

Moment or torque:

The units of moment or torque are the units of energy.

Pressure:

1 pascal (Pa) = 1 Slu of pressure = $1\ \text{N}\cdot\text{m}^{-2}$
 = 10^{-5} bar = $(10^{-5}/c_p)$ atmosphere (atm)
 = $(0.1/c_p)$ kgf $\cdot\text{m}^{-2}$ = 10 dyne $\cdot\text{cm}^{-2}$
 = $(0.0076/c_p)$ torr $\approx (1/133.322)$ torr
 $\approx (1/3376.85)$ inch of Hg at $60\ ^\circ\text{F}$ and g_n
 $\approx (1/248.84)$ inch of H₂O at $60\ ^\circ\text{F}$ and g_n
 $\approx 0.000\ 145\ 038$ lbf/in² (psi)

1 torr = 1 mm of Hg at $0\ ^\circ\text{C}$ and g_n
 $= (c_p/0.76)$ millibar (mbar) ≈ 1.33322 mbar

1 atmosphere = 1 standard atmosphere = 760 mm of Hg
 at $0\ ^\circ\text{C}$ and g_n = 760 torr = $10^5\ c_p$ Pa = c_p bar
 $\approx 14.695\ 949$ psi
 ≈ 30.0058 in of Hg at $60\ ^\circ\text{F}$ and g_n
 ≈ 33.9324 ft of H₂O at $60\ ^\circ\text{F}$ and g_n

1 in of Hg at $60\ ^\circ\text{F}$ ≈ 13.5702 in of H₂O at $60\ ^\circ\text{F}$

Viscosity**Viscosity (dynamic viscosity), η :**

1 poise (P) = $1\ \text{g}\cdot\text{cm}^{-1}\cdot\text{s}^{-1}$ = 0.1 Slu of η
 $= 0.1\ \text{kg}\cdot\text{m}^{-1}\cdot\text{s}^{-1}$

Kinematic viscosity, ν :

1 stoke (St) = $1\ \text{cm}^2\cdot\text{s}^{-1}$ = 10^{-4} Slu of ν = $10^{-4}\ \text{m}^2\cdot\text{s}^{-1}$

Temperature

1 Celsius (centigrade) degree ($^\circ\text{C}$) = 1.8 Fahrenheit degree ($^\circ\text{F}$)

1 degree Rankine ($^\circ\text{R}$) = 1.8 kelvin (K)
 (Note: no degree symbol is used with the symbol for kelvin.)

(Temperature in $^\circ\text{C}$) = (Temperature in $^\circ\text{F} - 32)/1.8$

Triple point of water = 273.16 K = 491.688 $^\circ\text{R}$

Energy and Power**Energy:**

1 joule (J) = 1 Slu of energy = 1 watt-second (W \cdot s)
 = $1\ \text{N}\cdot\text{m}$ = $1\ \text{Pa}\cdot\text{m}^3$ = 10^7 erg
 = $(1/4.184)$ gram calorie (gcal or cal)
 = $(1/4.1868)$ IT calorie or International Table calorie (cal_{IT})
 = $[(0.3048)^2\ c_m]^{-1}$ ft \cdot pdl
 $\approx (1/0.042\ 140\ 110)$ ft \cdot pdl
 $\approx (1/1055.06)$ British thermal unit (Btu)
 $\approx (1/1.355\ 817\ 9)$ ft \cdot lbf
 $\approx 4.587\ 338 \times 10^{17}$ Rydberg
 $\approx (10^{19}/1.60219)$ electron volt (eV)

1 Btu ≈ 778.172 ft \cdot lbf
 ≈ 25037 ft \cdot pdl
 ≈ 4414.37 cal

Energy of one quantum at wavelength λ cm
 $\approx (1.986\ 484 \times 10^{-23}/\lambda)$ J

Power:

1 watt (W) = 1 Slu of power = 1 J/s = $1\ \text{N}\cdot\text{m}/\text{s}$
 = $(1/746)$ electric horsepower (ehp)

1 mechanical horsepower (mhp or hp) = 550 ft \cdot lbf/s
 ≈ 745.700 W
 ≈ 0.99960 ehp

1 ton of refrigeration (U.S.) ≈ 3517 W
 ≈ 200 Btu/minute
 ≈ 840 cal/s
 ≈ 4.7143 ehp
 ≈ 4 frigorie

Luminous power: [Also see units of Illumination.]

1 lumen = $(1/683)$ watt of radiation at 540×10^{12} Hz
 (about 555 nm)

Quantity

1 mole (mol) = 0.001 kilomole (kmol) = amount that contains as many entities (molecules, atoms, ions, electrons, or other particles or groups of particles) as there are atoms in 12 grams of carbon 12.

When the entity is the molecule, unit *molecular weight* (traditional name) or *relative molar mass* (more precise name) = 1 gram per mol (g/mol) = 1 kilogram per kilomole (kg/kmol).

The number of molecules in 1 kilomole is $\approx 6.022\ 169 \times 10^{26}$ (Avogadro's number, N_A).

Radioactivity

Activity (source strength):

1 becquerel (Bq) = 1 nuclear disintegration or transformation per second = 1 s^{-1}
= 1 Slu of activity = 10^{-6} rutherford

1 curie (Ci) = 3.7×10^{10} Bq

Neutron fluence:

1 nvt = 1 neutron/cm² (unless another unit of area is specified)

Neutron flux (neutron flow rate):

1 nv = 1 neutron/(cm²•s) (unless other units of area or time are specified)

Radiation exposure:

1 coulomb per kilogram (C/kg) = 1 Slu of exposure

For X- or γ -radiation, 1 roentgen (R) produces 2.58×10^{-4} C/kg.

In air at STP (0 °C, 1 atm), 1 roentgen produces

1.610×10^{15} ion pairs per kg, 0.00838 J/kg,

6.76×10^{10} MeV/m³, 1 esu/cm³.

Absorbed dose of radiation (kerma, specific energy):

1 gray (Gy) = 1 J/kg = 1 Slu of absorbed dose = 100 rad or rd

Radiation dose equivalent (equivalent dose of radiation):

1 sievert (Sv) = 1 J/kg = 1 Slu of dose equivalent = 100 rem (roentgen equivalent mammal)

1 rep (roentgen equivalent physical) = dosage that produces 0.00838 J/kg in mammalian tissue

Electricity and Magnetism

Current, I:

1 ampere (A) = 1 Slu of $I = (c_0/10)$ statampere = 0.1 abampere

1 statampere = 1 Gu of $I = \sqrt{(4\pi)}$ HLu of I

Charge, Q:

1 coulomb (C) = 1 Slu of $Q = 1$ mksu of Q
= $(c_0/10)$ statcoulomb = 0.1 abcoulomb = $(1/96487)$ faraday

1 electron charge (e) = $1.602\ 1917 \times 10^{-19}$ C

Capacitance, C:

1 farad (F) = 1 Slu of $C = 1$ mksu of C
= $c_0^2 \times 10^{-9}$ statfarad = 10^{-9} abfarad

1 statfarad = 1 Gu of $C = 4\pi$ HLu of C

Permittivity, $\epsilon = \epsilon_r \epsilon_0$: [ϵ_r = relative permittivity (dimensionless)]

Permittivity of a vacuum, ϵ_0
= $[10^{11}/(4\pi c_0^2)]$ farad per meter (F/m), in SI
= $(10^{11}/c_0^2)$ F/m, in mks system = 1 statfarad/cm
= 1 Gu of $\epsilon = 1$ HLu of $\epsilon = (1/c_0^2)$ abfarad/cm

Resistance, R:

1 ohm (Ω) = 1 Slu of $R = 1$ mksu of R
= $(10^9/c_0^2)$ statohm = 10^9 abohm

1 statohm = 1 Gu of $R = [1/(4\pi)]$ HLu of R

Conductance, G:

1 siemens (S) = 1 Slu of $G = 1$ mksu of $G = (\text{ohm})^{-1}$
= $c_0^2 \times 10^{-9}$ esu of $G = 10^{-9}$ emu of G
= 1 mho (superseded name)

1 esu of $G = 1$ Gu of $G = 4\pi$ HLu of G

Resistivity, ρ_q :

1 ohm-meter ($\Omega \cdot \text{m}$) = 1 Slu of $\rho_q = 10^8$ microhm cm
= $[48 \times 10^8/(2.54\pi)]$ ohm per circular mil foot
= $(12 \times 10^8/2.54)$ ohm per square mil foot

Inductance, L:

1 henry (H) = 1 Slu of $L = 1$ mksu of $L = 10^9$ abhenry
= $(10^9/c_0^2)$ stathenry

1 stathenry = 1 Gu of $L = [1/(4\pi)]$ HLu of L

Electric flux, Ψ :

1 coulomb (C) = 1 Slu of $\Psi = 4\pi$ mksu of Ψ
= 0.4π emu of $\Psi = 0.4\pi c_0$ esu of Ψ

1 esu of $\Psi = 1$ Gu of $\Psi = [1/\sqrt{(4\pi)}]$ HLu of Ψ

Electric displacement, D:

1 coulomb per square meter (C/m²) = 1 Slu of D
= 4π mksu of $D = 4\pi c_0 \times 10^{-5}$ esu of D
= $4\pi \times 10^{-5}$ emu of D

1 esu of $D = 1$ Gu of $D = [1/\sqrt{(4\pi)}]$ HLu of D

Electric polarization, P:

1 coulomb per square meter (C/m²) = 1 Slu of P
= 1 mksu of $P = c_0 \times 10^{-5}$ statcoulomb/cm²
= 10^{-5} abcoulomb/cm²

1 statcoulomb/cm² = 1 Gu of $P = \sqrt{(4\pi)}$ HLu of P

Electric susceptibility per unit volume, $\chi_{e,v}$:

1 Slu of $\chi_{e,v} = [1/(4\pi)]$ esu of $\chi_{e,v} = 1$ HLu of $\chi_{e,v}$

1 esu of $\chi_{e,v} = 1$ mksu of $\chi_{e,v} = 1$ emu of $\chi_{e,v}$
= 1 Gu of $\chi_{e,v}$

Electric moment, M_q :

1 coulomb-meter (C•m) = 1 Slu of $M_q = 1$ mksu of M_q
= $10 c_0$ esu of $M_q = 10$ emu of M_q

1 esu of $M_q = 1$ Gu of $M_q = \sqrt{(4\pi)}$ HLu of M_q

Electromotive force (electric potential difference), E:

1 volt (V) = 1 Slu of $E = 1$ mksu of E
= $(10^8/c_0)$ statvolt = 10^8 abvolt

1 statvolt = 1 Gu of $E = [1/\sqrt{(4\pi)}]$ HLu of E

Electric field strength, \mathcal{E} :

1 volt per meter (V/m) = 1 Slu of \mathcal{E} = 1 mksu of \mathcal{E}
= $(10^6/c_0)$ statvolt/cm = 10^6 abvolt/cm

1 statvolt/cm = 1 Gu of \mathcal{E} = $[1/\sqrt{(4\pi)}]$ HLu of \mathcal{E}

Permeability, $\mu = \mu_r\mu_0$: [μ_r = relative permeability (dimensionless)]

Permeability of a vacuum, μ_0
= $4\pi \times 10^{-7}$ henry per meter (H/m) in SI
= 10^{-7} H/m in mks system = 1 abhenry/cm
= $(1/c_0^2)$ stathenry/cm = 1 Gu of μ = 1 HLu of μ

Magnetic pole strength, m^ :*

1 weber (Wb) = 1 Slu of m^* = 1 volt-second
= 1 ampere-henry = $(4\pi)^{-1}$ mksu of m^*
= $[10^8/(4\pi c_0)]$ statpole = $[10^8/(4\pi)]$ abpole

1 abpole = 1 Gu of m^* = $\sqrt{(4\pi)}$ HLu of m^* = $(1/c_0)$ statpole

Magnetic flux, Φ :

1 weber (Wb) = 1 Slu of Φ = 1 volt-second
= 1 ampere-henry = 10^8 maxwell (Mx) = 1 mksu of Φ
= $(10^8/c_0)$ esu of Φ = 10^8 emu of Φ

1 emu of Φ = 1 maxwell (Mx) = 1 line = 1 Gu of Φ
= $[1/\sqrt{(4\pi)}]$ HLu of Φ = $(1/c_0)$ esu of Φ

Magnetic induction, B :

1 tesla (T) = 1 Wb/m² = 1 Slu of B = 1 mksu of B
= 10^4 gauss = 10^4 Mx/cm²

1 gauss = 1 emu of B = $(1/c_0)$ esu of B = 1 Gu of B
= $[1/\sqrt{(4\pi)}]$ HLu of B

Magnetic polarization, J :

1 tesla (T) = 1 Wb/m² = 1 Slu of J
= $[1/(4\pi)]$ mksu of J = $[10^4/(4\pi)]$ emu of J = 10^9 gamma (γ)

1 emu of J = $(1/c_0)$ esu of J = 1 Gu of J = $\sqrt{(4\pi)}$ HLu of J

Magnetic susceptibility per unit volume, $\chi_{m,v}$:

1 Slu of $\chi_{m,v}$ = 1 HLu of $\chi_{m,v}$ = $[1/(4\pi)]$ emu of $\chi_{m,v}$

1 emu of $\chi_{m,v}$ = 1 esu of $\chi_{m,v}$ = 1 mksu of $\chi_{m,v}$
= 1 Gu of $\chi_{m,v}$

Magnetic moment, M_m :

1 weber-meter (Wb·m) = 1 Slu of M_m = $[1/(4\pi)]$ mksu of M_m
= $[10^{10}/(4\pi)]$ emu of M_m

1 emu of M_m = 1 Gu of M_m = $\sqrt{(4\pi)}$ HLu of M_m
= $(1/c_0)$ esu of M_m

Magnetomotive force, \mathcal{F} :

1 ampere-turn = 1 A = 1 Slu of \mathcal{F} = 4π mksu of \mathcal{F} = 0.4π emu of \mathcal{F}

1 gilbert = 1 emu of \mathcal{F} = c_0 esu of \mathcal{F} = 1 Gu of \mathcal{F}
= $[1/\sqrt{(4\pi)}]$ HLu of \mathcal{F}

Magnetic field strength, H :

1 ampere per meter (A/m) = 1 Slu of H = 4π mksu of H
= 0.004π oersted (Oe)

1 oersted (1 gauss before 1930) = 1 emu of H
= 1 Gu of H = c_0 esu of H = $[1/\sqrt{(4\pi)}]$ HLu of H

Reluctance, \mathcal{R} :

1 ampere-turn per weber = 1 Slu of \mathcal{R} = 4π mksu of \mathcal{R}
= $4\pi \times 10^{-9}$ emu of \mathcal{R} = $10^9/(4\pi)$ gilbert per maxwell

1 gilbert per maxwell = 1 emu of \mathcal{R} = c_0^2 esu of \mathcal{R}
= 1 Gu of \mathcal{R} = 1 HLu of \mathcal{R}

Radiation

Physical quantities in this field are measured in units of the dimensions indicated in Table 1-5; namely, units of energy, power, area, volume, time, and solid angle.

Illumination

Luminous intensity, I_v :

1 candela (cd) = 1 Slu of I_v = luminous intensity of a source that emits monochromatic energy of frequency 540×10^{12} Hz (about 555 nm wavelength) and that has a radiant intensity of $(1/683)$ watt per steradian = 1 candle

Luminous flux (luminous power), Φ_v :

1 lumen (lm) = 1 Slu of Φ_v = luminous flux emitted per steradian by a 1-candela source = 1 cd·sr

Luminance (of a source), L_v :

1 cd/m² = 1 Slu of L_v = 1 nit (nt) = 10^{-4} stilb (sb)
= π apostilb (asb) = $10^{-4}\pi$ lambert (L)

1 apostilb (asb) = 10^{-4} lambert (L) = $(0.3048)^2$ footlambert

Luminous flux density (of, through, or on a surface):

1 lumen per square meter (lm/m²)
= 1 Slu of luminous flux density

Luminous flux density, M_v , of a surface (luminous exitance, luminous emittance):

1 lm/m² = 1 Slu of M_v = $(0.3048)^2$ lm/ft²

Luminous flux density, E_v , on a surface (illuminance, illumination):

1 lm/m² = 1 lux = 1 Slu of E_v = 10^{-4} phot (ph)
= $(0.3048)^2$ footcandle

1 footcandle = 1 lm/ft²

Luminous energy, W_v :

1 lumen-second (lm·s) = 1 talbot = 1 Slu of W_v

Luminous efficacy, K_v :

1 lumen per watt (lm/W) = 1 Slu of K_v

Maximum luminous efficacy, $K_{v,max}$ = $(1/683)$ lm/W

Sound

For psychoacoustical units, see Example 2.8h.

TABLE 2-2.—SOME RELATIONSHIPS AMONG COHERENT UNITS

$1 \text{ N} = 1 \text{ kg}\cdot\text{m}/\text{s}^2 = 1 \text{ Wb}\cdot\text{A}/\text{m} = 1 \text{ Wb}^2/(\text{H}\cdot\text{m}) = 1 \text{ C}\cdot\text{V}/\text{m} = 1 \text{ C}^2/(\text{F}\cdot\text{m})$																																																				
$1 \text{ dyne} = 1 \text{ g}\cdot\text{m}/\text{s}^2 = 1 (\text{statcoulomb})^2/(\text{cm}\cdot\text{stathenry}) = 1 (\text{abcoulomb})^2/(\text{cm}\cdot\text{abhenry})$																																																				
$1 \text{ pdl} = 1 \text{ lb}\cdot\text{ft}/\text{s}^2$																																																				
$1 \text{ lbf} = 1 \text{ slug}\cdot\text{ft}/\text{s}^2 = g_n\cdot 1 \text{ lbm}$																																																				
$1 \text{ gf} = g_n\cdot 1 \text{ g}$																																																				
$1 \text{ J} = 1 \text{ W}\cdot\text{s} = 1 \text{ N}\cdot\text{m} = 1 \text{ V}\cdot\text{C} = 1 \text{ A}\cdot\text{Wb} = 1 \text{ Pa}\cdot\text{m}^3$																																																				
$1 \text{ erg} = 1 \text{ dyne}\cdot\text{cm} = 1 \text{ statvolt}\cdot\text{statcoulomb} = 1 \text{ abfarad}\cdot\text{abcoulomb}$																																																				
$1 \text{ second} = 1 \text{ F}\cdot\Omega = 1 \text{ statfarad}\cdot\text{stathm} = 1 \text{ abfarad}\cdot\text{abohm}$ $\quad = 1 (\text{HLu of C})\cdot(\text{HLu of R})$ $\quad = 1 \text{ H}/\Omega = 1 \text{ stathenry}/\text{stathm} = 1 \text{ abhenry}/\text{abohm} = 1 (\text{HLu of L})/(\text{HLu of R})$ $\quad = 1 \text{ Wb}/\text{V} = 1 \text{ C}/\text{A}$																																																				
$1 \text{ F} = 1 \text{ C}/\text{V}; 1 \text{ statfarad} = 1 \text{ statcoulomb}/\text{statvolt};$ $\quad 1 \text{ abfarad} = 1 \text{ abcoulomb}/\text{abvolt}; 1 (\text{HLu of C}) = 1 (\text{HLu of Q})/(\text{HLu of E})$																																																				
$1 \text{ H} = 1 \text{ Wb}/\text{A}; 1 \text{ stathenry} = 1 (\text{esu of } \Phi)/\text{statampere}; 1 \text{ abhenry} = 1 (\text{emu of } \Phi)/\text{abampere}$ $1 (\text{HLu of L}) = c_0 (\text{HLu of } \Phi)/(\text{HLu of I})$																																																				
<p>Symbols for units used in this table:</p> <table border="0"> <tr> <td>A</td><td>ampere</td><td>g</td><td>gram mass</td><td>lb</td><td>pound mass</td><td>pd</td><td>poundal</td> </tr> <tr> <td>C</td><td>coulomb</td><td>gf</td><td>gram force</td><td>lbf</td><td>pound force</td><td>s</td><td>second</td> </tr> <tr> <td>cm</td><td>centimeter</td><td>H</td><td>henry</td><td>m</td><td>meter</td><td>V</td><td>volt</td> </tr> <tr> <td>F</td><td>farad</td><td>J</td><td>joule</td><td>N</td><td>newton</td><td>W</td><td>watt</td> </tr> <tr> <td>ft</td><td>foot</td><td>kg</td><td>kilogram mass</td><td>Pa</td><td>pascal</td><td>Wb</td><td>weber</td> </tr> <tr> <td>Ω</td><td>ohm</td><td></td><td></td><td></td><td></td><td></td><td></td> </tr> </table>					A	ampere	g	gram mass	lb	pound mass	pd	poundal	C	coulomb	gf	gram force	lbf	pound force	s	second	cm	centimeter	H	henry	m	meter	V	volt	F	farad	J	joule	N	newton	W	watt	ft	foot	kg	kilogram mass	Pa	pascal	Wb	weber	Ω	ohm						
A	ampere	g	gram mass	lb	pound mass	pd	poundal																																													
C	coulomb	gf	gram force	lbf	pound force	s	second																																													
cm	centimeter	H	henry	m	meter	V	volt																																													
F	farad	J	joule	N	newton	W	watt																																													
ft	foot	kg	kilogram mass	Pa	pascal	Wb	weber																																													
Ω	ohm																																																			
<p>Constants used in this table:</p> $c_0 = 2.99792458 \times 10^{10}$ $g_n = (10 c_s)\text{m}/\text{s}^2 = (c_s/0.03048) \text{ft}/\text{s}^2 \approx 32.1740 \text{ft}/\text{s}^2$ $c_s = 0.980665$																																																				

TABLE 2-3.—PREFIXES OF THE SYSTÈME INTERNATIONAL

Power of 10	Prefix	Symbol	Power of 10	Prefix	Symbol
(1)	deka	da	(-1)	deci	d
(2)	hecto	h	(-2)	centi	c
3	kilo	k	-3	milli	m
6	mega	M	-6	micro	μ
9	giga	G	-9	nano	n
12	tera	T	-12	pico	p
15	peta	P	-15	femto	f
18	exa	E	-18	atto	a

Note: The prefixes enclosed in parentheses are less desirable. Powers that are multiples of 3 are preferred.

TABLE 5-1.—STATION LOCATIONS AND WEIGHTS FOR AVERAGING

Averaging for linear interval $0 \leq x \leq 1$
 Averaging in a circular duct, in interval $0 \leq r \leq 1$

Number of stations <i>n</i>	Method											
	Centroid of equal areas ^(a)			Newton-Cotes			Chebyshev ^(a)			Gauss		
	<i>x</i>	<i>r</i>	<i>w</i>	<i>x</i>	<i>r</i>	<i>w</i>	<i>x</i>	<i>r</i>	<i>w</i>	<i>x</i>	<i>r</i>	<i>w</i>
2	0.2500	0.5000	1/2	(b)	(b)	(b)	(e)	(e)	(e)	(e)	(e)	(e)
	.7500	.8660		0	0	1/2	0.2113	0.4597	1/2	0.2113	0.4597	1/2
3	.1667	0.4082	1/3	(c)	(c)	(c)	(e)	(e)	1/3	(e)	(e)	(e)
	.5000	.7071		0	0	0.1667	0.1464	0.3827		0.1127	0.3357	0.2778
	.8333	.9129		0.5	0.7071	.6667	.5000	.7071		.5000	.7071	.4444
4	.1250	0.3536	1/4	(d)	(d)	(d)	(e)	(e)	1/4	(e)	(e)	(e)
	.3750	.6124		0	0	0.1250	0.1027	0.3203		0.0694	0.2635	0.1739
	.6250	.7906		0.3333	0.5774	.3750	.4062	.6373		.3300	.5745	.3261
	.8750	.9354		.6667	.8165	.3750	.5938	.7706		.6700	.8185	.3261
5	.1000	0.3162	1/5	0	0	0.0778	0.0838	0.2894	1/5	0.0469	0.2166	0.1185
	.3000	.5477		0.25	0.5000	.3556	.3127	.5592		.2308	.4804	.2393
	.5000	.7071		.50	.7071	.1333	.5000	.7071		.5000	.7071	.2844
	.7000	.8367		.75	.8660	.3556	.6873	.8290		.7692	.8771	.2393
	.9000	.9487		1	1	.0778	.9162	.9572		.9531	.9763	.1185
6	0.0833	0.2887	1/6	0	0	0.0660	0.0669	0.2586	1/6	0.0338	0.1838	0.0857
	.2500	.5000		0.2	0.4472	.2604	.2887	.5373		.1694	.4116	.1804
	.4167	.6455		.4	.6325	.1736	.3667	.6055		.3807	.6170	.2340
	.5833	.7638		.6	.7746	.1736	.6333	.7958		.6193	.7870	.2340
	.7500	.8660		.8	.8944	.2604	.7113	.8434		.8306	.9114	.1804
	.9167	.9574		1	1	.0660	.9331	.9660		.9662	.9830	.0857
7	0.0714	0.2673	1/7	0	0	0.0488	0.0581	0.2410	1/7	0.0254	0.1595	0.0647
	.2143	.4629		0.1667	0.4082	.2571	.2352	.4849		.1292	.3595	.1399
	.3571	.5976		.3333	.5774	.0321	.3381	.5814		.2971	.5450	.1909
	.5000	.7071		.5000	.7071	.3238	.5000	.7071		.5000	.7071	.2090
	.6429	.8018		.6667	.8165	.0321	.6619	.8136		.7029	.8384	.1909
	.7857	.8864		.8333	.9129	.2571	.7648	.8745		.8708	.9331	.1399
	.9286	.9636		1	1	.0488	.9419	.9705		.9746	.9872	.0647
<i>n</i>	<i>x</i>	<i>r</i>	<i>w</i>	<i>x</i>	<i>r</i>	<i>w</i>	<i>x</i>	<i>r</i>	<i>w</i>	<i>x</i>	<i>r</i>	<i>w</i>
	Centroid of equal areas			Newton-Cotes			Chebyshev			Gauss		

TABLE 5-1.—Concluded.

Number of stations <i>n</i>	Method											
	Centroid of equal areas ^(a)			Newton-Cotes			Chebyshev ^(a)			Gauss		
	<i>x</i>	<i>r</i>	<i>w</i>	<i>x</i>	<i>r</i>	<i>w</i>	<i>x</i>	<i>r</i>	<i>w</i>	<i>x</i>	<i>r</i>	<i>w</i>
8	0.0625	0.2500	1/8	0	0	0.0435	(f)	(f)	(f)	0.0199	0.1409	0.0506
	.1875	.4330		0.1429	0.3780	.2070	.2031	.4507	.1017	.3189	.1112	
	.3125	.5590		.2857	.5345	.0766	.2969	.5449	.2372	.4871	.1569	
	.4375	.6614		.4286	.6547	.1730	.4487	.6698	.4083	.6390	.1813	
	.5625	.7500		.5714	.7559	.1730	.5513	.7425	.5917	.7692	.1813	
	.6875	.8292		.7143	.8452	.0766	.7031	.8385	.7628	.8734	.1569	
	.8125	.9014		.8571	.9258	.2070	.7969	.8927	.8983	.9478	.1112	
	.9375	.9682		1	1	.0435	.9487	.9740	.9801	.9900	.0506	
9	0.0556	0.2357	1/9	0	0	0.0349	0.0442	0.2103	1/9	0.0159	0.1262	0.0406
	.1667	.4082		0.1250	0.3536	.2077	.1995	.4466		.0820	.2863	.0903
	.2778	.5270		.2500	.5000	-.0327	.2356	.4854		.1933	.4397	.1303
	.3889	.6236		.3750	.6124	.3702	.4160	.6450		.3379	.5813	.1562
	.5000	.7071		.5000	.7071	-.1601	.5000	.7071		.5000	.7071	.1651
	.6111	.7817		.6250	.7906	.3702	.5840	.7642		.6621	.8137	.1562
	.7222	.8498		.7500	.8660	-.0327	.7644	.8743		.8067	.8982	.1303
	.8333	.9129		.8750	.9354	.2077	.8005	.8947		.9180	.9581	.0903
	.9444	.9718		1	1	.0349	.9558	.9776		.9841	.9920	.0406
10	0.05	0.2236	1/10	0	0	0.0319	(g)	(g)	(g)	0.0130	0.1142	0.0333
	.15	.3873		0.1111	0.3333	.1757	.1564	.3954	.0674	.2597	.0747	
	.25	.5000		.2222	.4714	.0121	.2500	.5000	.1603	.4004	.1095	
	.35	.5916		.3333	.5774	.2159	.3436	.5862	.2833	.5323	.1346	
	.45	.6708		.4444	.6667	.0645	.4581	.6768	.4256	.6524	.1478	
	.55	.7416		.5556	.7454	.0645	.5419	.7361	.5744	.7579	.1478	
	.65	.8062		.6667	.8165	.2159	.6564	.8102	.7167	.8466	.1346	
	.75	.8660		.7778	.8819	.0121	.7500	.8660	.8397	.9164	.1095	
	.85	.9220		.8889	.9428	.1757	.8436	.9185	.9325	.9657	.0747	
.95	.9747	1	1	.0319	.9581	.9788	.9870	.9935	.0333			
<i>n</i>	<i>x</i>	<i>r</i>	<i>w</i>	<i>x</i>	<i>r</i>	<i>w</i>	<i>x</i>	<i>r</i>	<i>w</i>	<i>x</i>	<i>r</i>	<i>w</i>
	Centroid of equal areas			Newton-Cotes			Chebyshev			Gauss		
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(a) All measurements of equal weight						(e) 0.2–0.8 rule						
(b) Trapezoidal rule						(f) Two 4-station intervals						
(c) Parabolic rule (Simpson's rule)						(g) Two 5-station intervals						
(d) Three-eighths rule												

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1. Report No. NASA RP-1222		2. Government Accession No.		3. Recipient's Catalog No.	
4. Title and Subtitle Foundations of Measurement and Instrumentation				5. Report Date April 1990	
				6. Performing Organization Code	
7. Author(s) Isidore Warshawsky				8. Performing Organization Report No. E-3786	
				10. Work Unit No. 505-62-01	
9. Performing Organization Name and Address National Aeronautics and Space Administration Lewis Research Center Cleveland, Ohio 44135-3191				11. Contract or Grant No.	
				13. Type of Report and Period Covered Reference Publication	
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546-0001				14. Sponsoring Agency Code	
15. Supplementary Notes					
16. Abstract This text is intended to provide to the user of instrumentation an understanding of the factors that influence instrument performance, selection, and application, and of the methods of interpreting and presenting the results of measurements. Such understanding is prerequisite to the successful attainment of the best compromise among reliability, accuracy, speed, cost, and importance of the measurement operation in achieving the ultimate goal of a project. Some subjects covered are dimensions; units; sources of measurement error; methods of describing and estimating accuracy; deduction and presentation of results through empirical equations, including the method of least squares; experimental and analytical methods of determining the static and dynamic behavior of instrumentation systems, including the use of analogs.					
17. Key Words (Suggested by Author(s)) Measurement; Instrumentation; Dimensions; Units; Accuracy of measurement; Curve fitting; Empirical equations; Errors of measurement; Dynamic response; Systems measurement; Instrument performance; Instrument selection				18. Distribution Statement Unclassified - Unlimited Subject Category 35	
19. Security Classif. (of this report) Unclassified		20. Security Classif. (of this page) Unclassified		21. No of pages 240	22. Price* A11

